A New Approach to the Duplication of Blocks by a Vertex and AUM Block Mean Labeling with an Application in Water Management

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KEYWORDS

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ABSTRACT

AUM block mean labeling is constructed from the labels of the vertices and edges. It is an interesting and rapidly developing labeling technique with a wide scope for applications in different fields. Duplication of vertices and edges is another direction of research among graph labeling techniques. In the present work, the notion of duplication of a block by a vertex and duplication of multiple blocks by a vertex are introduced and further discussed for Path P_n. Duplication of a block by a vertex means a new vertex is appended to the graph, and that vertex is joined to the vertices that are common to that block and its neighbouring blocks. Also, AUM block mean labeling is investigated for some graphs that are obtained by duplication of a block by a vertex. The concept of duplication of blocks by a vertex finds its application in water management. A new algorithm is presented by applying the concept of duplication of blocks, to facilitate a drainage system and water supply without blockages in this paper

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1.Introduction

Graph labeling refers to the process of assigning labels or values to the vertices or edges of a graph. It is a vital concept in graph theory initiated by Rosa. A [3]. In [7] and [8], A. Uma Maheswari et al. devised new techniques for labeling the blocks of a graph, known as AUM block sum labeling, AUM block labeling. These techniques were further studied for Star, Bistar, Comb, Sunlet, Diamond snake, Pentagonal snake, Alternate pentagonal snake, Perfect binary tree, Triangular snake, Friendship, tadpole, Cactus, Kite graphs in [7], [8], [9], [10], [11], [12] and [14]. AUM block mean labeling, which is assigned by averaging the labels on the vertices and edges, is a new variation of block labeling that the authors developed in [17].

AUM block mean labeling for the Snake graph families and Chemical graphs, as well as Path graph and families of Star graphs, was established in [17] and [18]. AUM block sum labeling has been used for cryptographic techniques to ensure safe communication in [13], [15], and [16]. In [18], an algorithm is devised for optimal tool assignment in a company for industrial application using AUM block mean labeling.

The concept of duplication of graph elements was introduced by S. K. Vaidya et al. Further it is studied and discussed in [20], [21]. In [6], the Parity combination labeling of graphs that were duplicated by graph elements of Path, Cycle, and Star graphs was discussed. The Group S_3 remainder labeling was investigated for duplication of graph elements in the Star graph [5]. In [19], AUM block mean labeling was discussed for the graphs, which are obtained by the duplication of graph elements for Path and Star graphs.

The concept of duplication of graph elements motivated the authors to introduce the new concept of duplication of a block by a vertex and duplication of multiple blocks by a vertex. This concept is used in real-life situations, such as water supply, drainage connection, cable connection, power grids and so on. The concept is applied to Path graph and given in Section 3. In Section 4, AUM block mean labeling is established for the graphs that are constructed by duplication of a block by a vertex for Path, Middle graph of Path, Triangular snake, and Alternate triangular snake graphs. In Section 5, an algorithm is given to enhance the efficiency and reliability of water management and explained with an illustration.

2.Preliminaries

In the following section, basic definitions and results required for the study of this paper are listed out:

Definition 2.1: AUM block mean labeling [17]

Let G be a graph with p vertices, a set of vertices called $V(G)$, q edges, a set of edges called $E(G)$, and r blocks, a set of blocks called $B(G)$, $p,q,r \geq 1$.

We say that the graph G admits **AUM block mean labeling** if there exists a bijection $\phi: V(G) \rightarrow \{1,3,5, \dots 2p-1\}$ and an injective function $\phi^* : E(G) \rightarrow \{2,3,4, \dots 2p-2\}$ induced from ϕ by $\phi^*(uv) = \frac{\phi(u)+\phi(v)}{2}$ and $\phi^{**}: B(G) \rightarrow Z^+$ defined as follows:

Let B_k be incident with the vertices $v_{k_1}, v_{k_2}, \dots \dots v_{k_m}, 1 \leq k_m \leq p$ and edges $e_{k_1}, e_{k_2}, \dots \dots e_{k_n}$ $1 \leq k_n \leq q$.

Define $\phi^{**}(B_k) = \frac{\sum_{i=1}^{m} \phi(v_{k_i}) + \sum_{i=1}^{n} \phi^*(e_{k_i})}{No.of\ vertices\ incident\ with\ B_k + No.of\ edges\ incident\ with\ B_k}$ and

$\phi^{**}(B_k) \neq \phi^{**}(B_j)$ for $1 \leq k, j \leq r$ and $k \neq j$.

A graph G is known as an **AUM block mean labelled graph** if it permits AUM block mean labeling.

Definition 2.2: Duplication of a vertex [22]

A new graph G' is produced when a vertex u of a graph G is duplicated by adding a new vertex u' such that $N(u') = N(u)$.

Definition 2.3: Duplication of an edge by a vertex [20]

Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.

Definition 2.4: Triangular snake graph [4]

The Triangular snake T_n is a graph that is formed by adding a triangle C_3 to each edge of the Path graph P_n.

Definition 2.5: Alternate triangular snake graph [2]

An Alternate triangular snake $A(T_n)$ is obtained by every alternate edge of a Path being replaced by C_3.

Definition 2.6: Middle graph [1]

Let G be a graph with a vertex set V and an edge set E. The middle graph of G, denoted by $M(G)$, is the graph whose vertex set is $V \cup E$.

Two vertices x and y in the vertex set of $M(G)$ are adjacent in $M(G)$ if:

(1) x and y are adjacent edges in G or

(2) $x \in V$, $y \in E$ and x, y are incident in G.

Theorem 2.7: [19]

The graph obtained by duplication of an edge in P_n by a vertex is an AUM block mean labelled graph.

Theorem 2.8: [18]

Every Path $P_n, n \geq 2$ is an AUM block mean labelled graph.

3. Duplication of blocks by a vertex

In this section, the new concept of duplication of a block by a vertex and duplication of multiple blocks by a vertex in a graph is defined and explained for Path graph.

Definition 3.1: Neighbourhood of a block:

Let G be any (nontrivial, finite) graph. The neighbourhood of a block B of G is the set of all blocks that have a common vertex with B, and it is denoted by $N(B)$.

Definition 3.2: Duplication of a block by a vertex

Let G be any graph. Duplication of a block B in G by a new vertex w is the graph, which is obtained by joining the vertex w with the vertices, which are common to B and its neighbouring blocks. The graph after duplication of a block B by a vertex is denoted by $D_B(G)$.

Proposition 3.3: Duplication of a block by a vertex in the Path $P_n (n \geq 2)$

Consider a Path graph P_n with the vertices $v_1, v_2, \ldots\ldots v_n$, and the blocks $B_1, B_2, \ldots\ldots B_{n-1}$. When $n = 2$, the block B_1 has no neighbouring blocks. Hence, if the block B_1 is duplicated by a new vertex w then $D_{B_1}(P_2)$ is a disconnected graph [Refer to Figure: 1].

Figure 1: Graph P_2 and $D_{B_1}(P_2)$

When $n = 3$, the block B_1 has the neighbouring block B_2 and B_2 has the neighbouring block B_1. Hence, the duplicated graph $D_{B_1}(G)$ and $D_{B_2}(G)$ by duplicating a new vertex w is obtained by joining the vertex w to the vertex v_2 [Refer to Figure: 2].

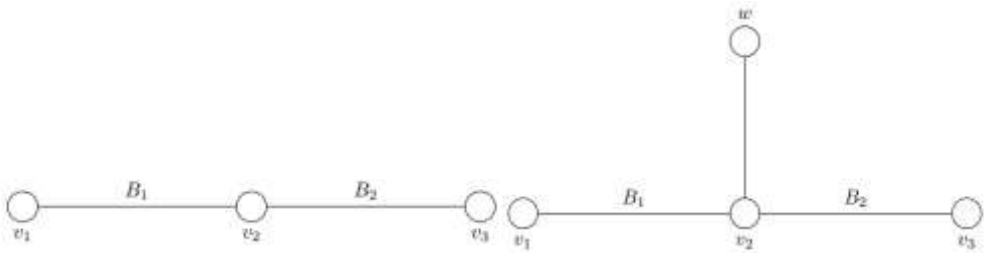


Figure 2: Graph P_3 and $D_{B_1}(P_3)/D_{B_2}(P_3)$

When $n \geq 4$, the following are the possible cases for duplication of a block by a vertex w in the Path P_n:

Figure 3: Graph P_n

Here, $N(B_1) = \{B_2\}, N(B_i) = \{B_{i-1}, B_{i+1}\},\ 2 \leq i \leq n-2$ and $N(B_{n-1}) = \{B_{n-2}\}$.

Case (i): Duplication of the block B_1 in P_n

Consider the block B_1, which is duplicated by a new vertex w. In this case, B_2 is the neighbouring block of B_1 and the vertex v_2 is common to the blocks B_1 and B_2. Therefore, join the vertices w and v_2 to obtain the graph $D_{B_1}(P_n)$ [Refer to Figure: 4].

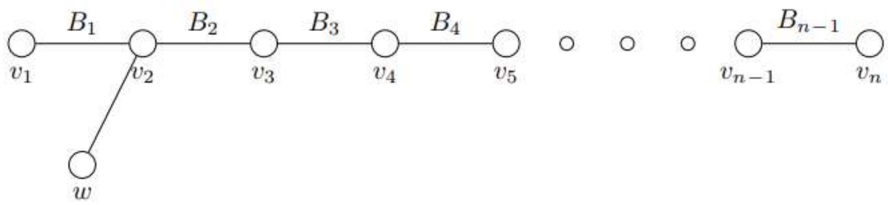


Figure 4: Graph $D_{B_1}(P_n)$

Case (ii): Duplication of the block B_{n-1} in P_n

Consider the block B_{n-1}, which is duplicated by a new vertex w. In this case, B_{n-2} is the only neighbouring block of B_{n-1} and the vertex v_{n-1} is common to the blocks B_{n-1} and B_{n-2}. Therefore, join the vertices w and v_{n-1} to obtain the graph $D_{B_{n-1}}(P_n)$ [Refer to Figure: 5].

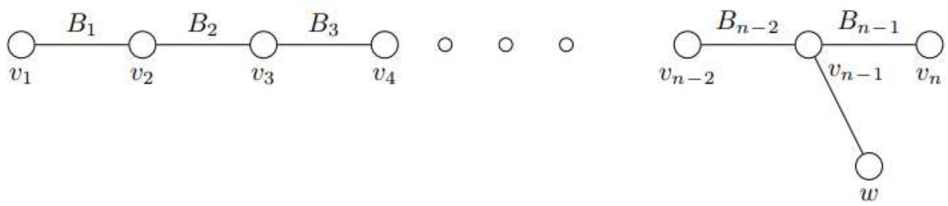


Figure 5: Graph $D_{B_{n-1}}(P_n)$

Case (iii): Duplication of the intermediate block B_i $(2 \leq i \leq n-2)$ in P_n

Consider the block B_i, which is duplicated by a new vertex w. In this case, B_{i-1} and B_{i+1} are the neighbouring blocks of B_i and the vertices v_i and v_{i+1} are common to the block B_i and its neighbouring blocks B_{i-1} and B_{i+1} respectively. Hence, join the vertices w to v_i and v_{i+1} to get the graph $D_{B_i}(P_n)$ [Refer to Figure: 6].

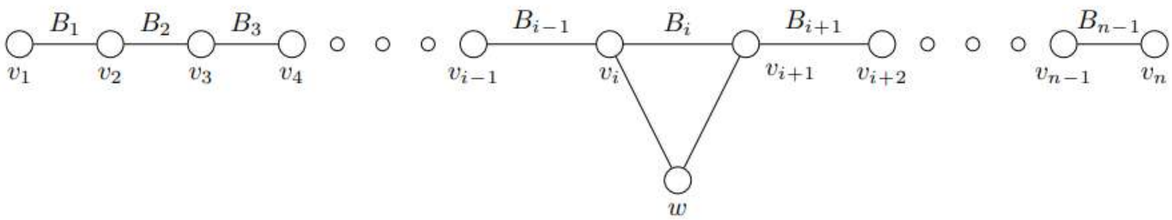


Figure 6: Graph $D_{B_i}(P_n)$

Definition 3.4: Duplication of multiple blocks by a vertex:

Let G be any graph, n be the number of blocks in the graph G. Duplication of m blocks ($2 \leq m \leq n$), by a vertex w means, each of the m blocks of the graph G is duplicated by a vertex w. The graph obtained by the duplication of m blocks, namely $B_1, B_2, \ldots B_m$ by a vertex of a graph G is denoted by $D^m_{B_1,B_2,\ldots B_m}(G)$.

In this study, duplication of 2 blocks by a vertex is considered.

Proposition 3.5: Duplication of 2 blocks by a vertex in the Path $P_n (n \geq 3)$

Consider a Path graph P_n with the vertices $v_1, v_2, \dots \dots v_n$, and the blocks $B_1, B_2, \dots \dots B_{n-1}$.

When $n = 3$, the block B_1 has the neighbouring block B_2 and B_2 has the neighbouring block B_1. Hence, the duplicated graph $D^2_{B_1,B_2}(P_3)$ by duplicating a new vertex w is obtained by joining the vertex w to the vertex v_2 [Refer to Figure: 7].

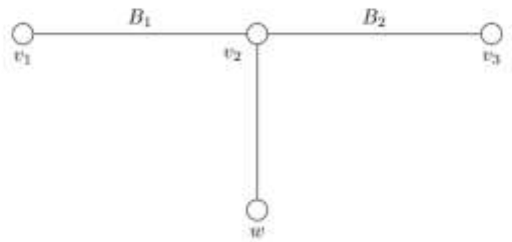


Figure 7: Graph $D^2_{B_1,B_2}(P_3)$

When $n \geq 4$, the following are the possible cases for duplication of 2 blocks by a vertex w in the Path P_n:

Case (i): Duplication of the blocks B_1 and B_2 in P_n

B_2 is the neighbouring block of B_1, join the vertices w and v_2, in which v_2 is common to B_1 and B_2.

Also, join w to v_2 and v_3, in which v_2 and v_3 are common to the block B_2 and its neighbouring blocks B_1 and B_3 respectively. But already w is joined to v_2, join w to v_3.

The graph $D^2_{B_1,B_2}(P_n)$ is given as follows:

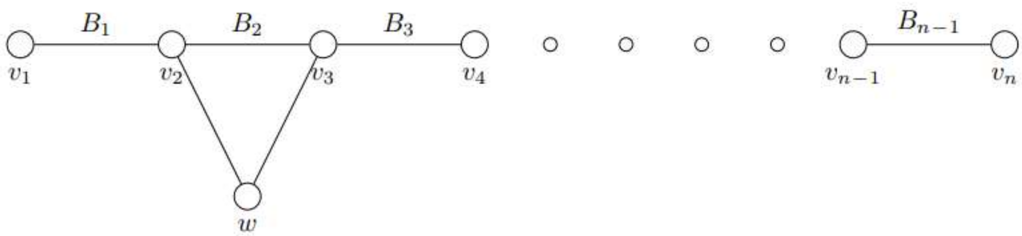


Figure 8: Graph $D^2_{B_1,B_2}(P_n)$

Case (ii): Duplication of the blocks B_1 and B_i $(3 \leq i \leq n-2)$ in P_n

Join the vertices w and v_2, in which v_2 is common to the block B_1 and its neighbouring block B_2 for duplication of the block B_1.

Also, B_{i-1} and B_{i+1} are the neighbouring blocks of B_i and the vertices v_i and v_{i+1} are common to the block B_i and its neighbouring blocks B_{i-1} and B_{i+1} respectively.

Therefore, join the vertices w to v_i and v_{i+1} to obtain the graph $D^2_{B_1,B_i}(P_n)$[Refer to Figure: 9].

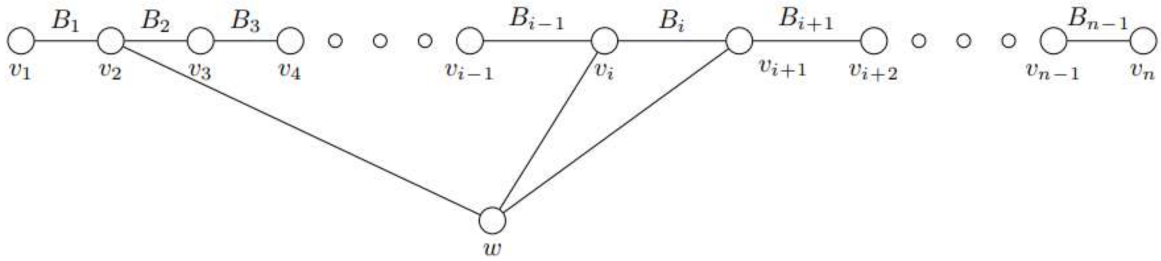


Figure 9: Graph $D^2_{B_1,B_i}(P_n)$

Case (iii): Duplication of the blocks B_1 and B_{n-1} in P_n

In this case, B_2 is the neighbouring block of B_1, connect the vertices w and v_2.

Also, B_{n-2} is the only neighbouring block of B_{n-1}, and the vertex v_{n-1} is common to B_{n-1} and B_{n-2}. Therefore, join v_{n-1} and w.

The graph $D^2_{B_1,B_{n-1}}(P_n)$ is given as follows:

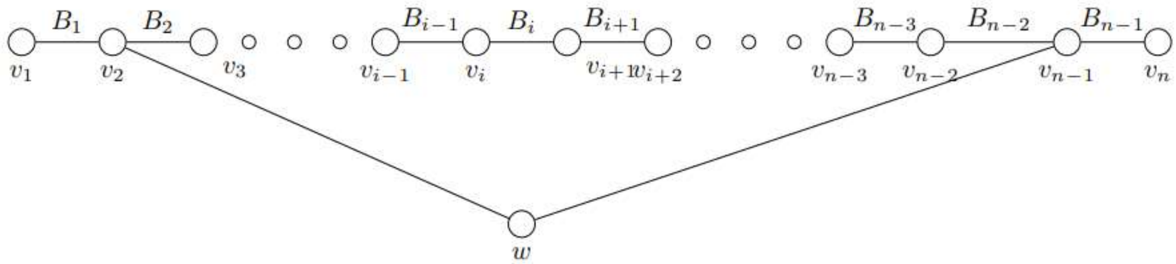


Figure 10: Graph $D^2_{B_1,B_{n-1}}(P_n)$

Case (iv): Duplication of the blocks B_i $(2 \le i \le n-2)$ and B_j $(i+1 \le j \le n-2)$ in P_n

B_i has the neighbourhood B_{i-1} and B_{i+1} and the vertices v_i and v_{i+1} are common to the block B_i and its neighbouring blocks B_{i-1} and B_{i+1} respectively.

Also, B_j has the neighbourhood B_{j-1} and B_{j+1} and the vertices v_j and v_{j+1} are common to the block B_j and its neighbouring blocks B_{j-1} and B_{j+1} respectively. [In the case $j = i+1$, the vertex $v_{i+1} = v_j$].

To obtain the duplicated graph $D^2_{B_i,B_j}(P_n)$, join the vertices w to v_i, v_{i+1}, v_j and v_{j+1}.

The graph $D^2_{B_i,B_j}(P_n)$ is given as follows:

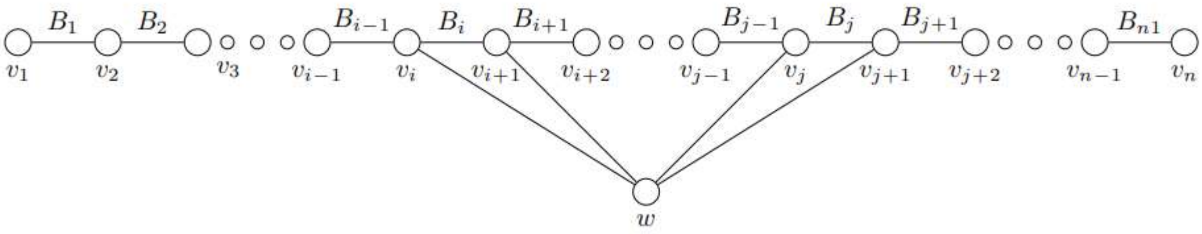


Figure 11: Graph $D^2_{B_i,B_j}(P_n)$

Case (v): Duplication of the blocks $B_i (2 \leq i \leq n-2)$ and B_{n-1} in P_n

B_i has the neighbourhood B_{i-1} and B_{i+1} and the vertices v_i and v_{i+1} are common to the block B_i and its neighbouring blocks B_{i-1} and B_{i+1} respectively.

Also, B_{n-1} has the only neighbourhood B_{n-2} and the vertex v_{n-1} is common to the block B_{n-1} and B_{n-2}.

To obtain the duplicated graph $D^2_{B_i,B_{n-1}}(P_n)$, join the vertices w to v_i, v_{i+1}, and v_{n-1}.

The graph $D^2_{B_i,B_{n-1}}(P_n)$ is given as follows:

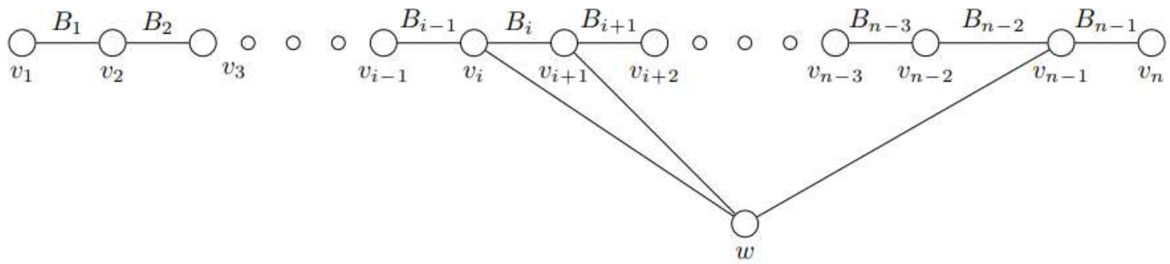


Figure 12: Graph $D^2_{B_i,B_{n-1}}(P_n)$

4. AUM block mean labeling for the duplicated graphs

AUM block mean labeling in block duplication provides a way to analyse and understand the graphs in various contexts. Here, AUM block mean labeling is investigated for the Middle graph of Path, Triangular snake, and Alternate triangular snake graphs. Also, the graphs constructed through duplication of a block by a vertex in the Path, Middle graph of Path, Triangular snake, and Alternate triangular snake graphs that admit AUM block mean labeling are discussed in this study.

Theorem 4.1: The graph $D_B(P_n)$, $n \geq 2$ is an AUM block mean labelled graph.

Proof: Case (i) $\boldsymbol{n = 2}$

Then P_2 has only one block. Duplication of that block by a vertex w is clearly admits AUM block mean labeling.

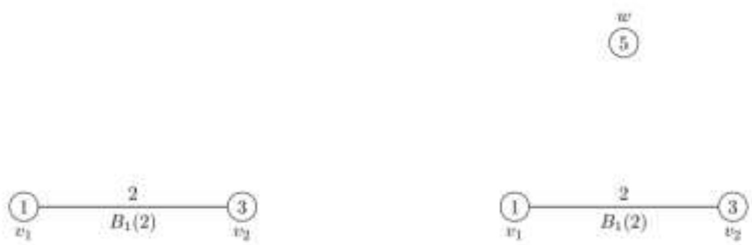


Figure 13: Path P_2 and AUM block mean labeling by duplication of a block B_1 in P_2 by a vertex w

Case (ii) $n \geq 3$

Let $D_{B_i}(P_n)$ be the graph produced through the duplication of a block B_i in P_n by a new vertex w.

Since in the Path graph, every edge is a block and $D_{B_i}(P_n)$ becomes the graph obtained by duplication of an edge by a vertex.

Therefore, by Theorem 2.7, the graph $D_{B_i}(P_n)$ is an AUM block mean labelled graph.

Theorem 4.2: The graph $M(P_n)$ is an AUM block mean labelled graph.

Proof: Case (i): $n = 2$

The graph $M(P_2)$ is P_3 and from Theorem 2.8, it was proved that any Path $P_n (n \geq 2)$ admits AUM block mean labeling.

Case (ii): $n \geq 3$

Consider G to be a Middle graph of Path $M(P_n)$ with the vertices $u_1, u_2, \ldots\ldots u_n$ (vertices in the Path P_n) and $v_1, v_2, \ldots\ldots v_{n-1}$ (vertices corresponding to the edges of P_n to obtain $M(P_n)$), edges $e_1, e_2, \ldots\ldots e_{3n-4}$ and blocks $B_1, B_2, \ldots\ldots B_n$.

Define the bijective function $\phi : V(G) \to \{1,3,5, \ldots 2(2n-1)-1\}$ by

$\phi(u_i) = 4i - 3 \;\; \forall\, i = 1,2, \ldots n,\;\; \phi(v_i) = 4i - 1 \;\; \forall\, i = 1,2, \ldots n-1$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \to \{2,3,4, \ldots 2(2n-1)-2\}$ is defined from ϕ by $\phi^*(e_i) = 4i - 2$ where $e_i = u_i v_i \;\forall\; i = 1,2, \ldots n-1$

$\phi^*(e_{n-1+i}) = 4i$ where $e_{n-1+i} = u_{i+1} v_i \;\forall\; i = 1,2, \ldots n-1$

$\phi^*(e_{2n-2+i}) = 4i + 1$ where $e_{2n-2+i} = v_i v_{i+1} \;\forall\; i = 1,2, \ldots n-2$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_1) = 2,\; \phi^{**}(B_{k+1}) = 4k + 1 \;\;\forall\, k = 1,2,3, \ldots n-2,\; \phi^{**}(B_n) = 4(n-1)$

For $k \neq j,\; \phi^{**}(B_k) \neq \phi^{**}(B_j), 1 \leq k, j \leq n$.

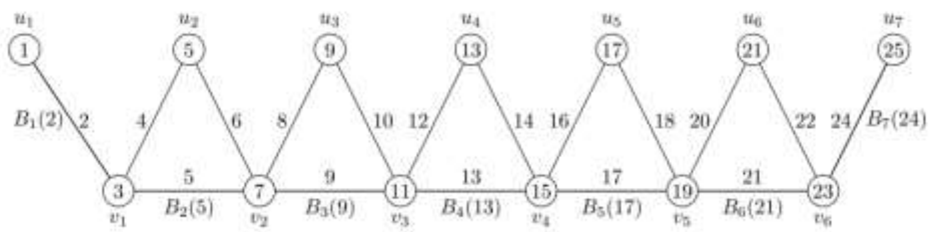


Figure 14: AUM block mean labeling of $M(P_7)$

Therefore, $M(P_n)$ is an AUM block mean labelled graph.

Theorem 4.3: The graph $D_B(M(P_n))$ is an AUM block mean labelled graph.

Proof: Consider a Middle graph of Path $M(P_n)$ with the vertices $u_1, u_2, \ldots\ldots u_n$ and $v_1, v_2, \ldots\ldots v_{n-1}$, edges $e_1, e_2, \ldots\ldots e_{3n-4}$ and blocks $B_1, B_2, \ldots\ldots B_n$.

Let $D_{B_j}(M(P_n))$ be the graph produced through the duplication of a block B_j in $M(P_n)$ by a new vertex w.

Consider the following different cases for block B_j, when B_j is at the origin or at the terminus, and when B_j is an intermediate block of $M(P_n)$.

Case 1: B_j is at the origin or at the terminus block of $M(P_n)$

Let us assume that B_j is at the origin block of $M(P_n)$.

Therefore, $B_j = B_1$ and we get an additional block B_1' after duplicating the block B_1 by a vertex w.

Also, $D_{B_1}(M(P_n)) = G$ has $2n$ vertices, $3n - 3$ edges and $n + 1$ blocks.

Define the bijective function $\phi : V(G) \to \{1,3,5, \ldots 2(2n) - 1\}$ by

$\phi(w) = 1,\ \phi(u_i) = 4i - 1\ \ \forall\ i = 1,2, \ldots n,\ \ \phi(v_i) = 4i + 1\ \ \forall\ i = 1,2, \ldots n - 1,$ assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \to \{2,3,4, \ldots 2(2n) - 2\}$ is defined from ϕ by

$\phi^*(e_1') = 3$ where $e_1' = wv_1$

$\phi^*(e_i) = 4i$ where $e_i = u_i v_i\ \forall\ i = 1,2, \ldots n - 1$

$\phi^*(e_{n-1+i}) = 4i + 2$ where $e_{n-1+i} = u_{i+1} v_i\ \forall\ i = 1,2, \ldots n - 1$

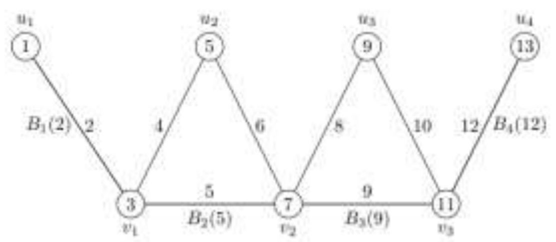
$\phi^*(e_{2n-2+i}) = 4i + 3$ where $e_{2n-2+i} = v_i v_{i+1}\ \forall\ i = 1,2, \ldots n - 2$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_1') = 3,\ \ \phi^{**}(B_1) = 4,$

$\phi^{**}(B_{k+1}) = 4k - 1\ \ \forall\ k = 1,2,3, \ldots n - 2,\ \ \phi^{**}(B_n) = 4(n - 1) + 2$



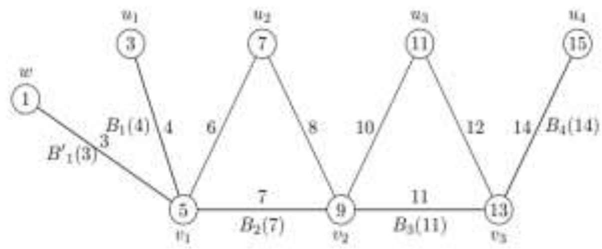


Figure 15: Middle graph of Path M(P_4) and AUM block mean labeling by duplication of a block B_1 in M(P_4) by a vertex w

Case 2: $B_j (2 \leq j \leq n-1)$ is an intermediate block of $M(P_n)$

Then $D_{B_j}(M(P_n)) = G$ has $2n$ vertices, $3n-2$ edges and n blocks.

Define the bijective function $\phi : V(G) \to \{1,3,5, \dots 2(2n)-1\}$ by

$\phi(u_i) = 4i-3 \;\forall\; i = 1,2, \dots j,\qquad \phi(u_{j+i}) = 4j-1+4i \quad \forall\; i = 1,2, \dots n-j$

$\phi(v_i) = 4i-1 \;\forall\; i = 1,2, \dots j-1,\quad \phi(v_{j-1+i}) = 4j-3+4i \;\forall\; i = 1,2, \dots n-j$

$\phi(w) = 4j-1$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \to \{2,3,4, \dots 2(2n)-2\}$ is defined from ϕ by

$\phi^*(e_i) = 4i-2$ where $e_i = u_i v_i \;\forall\; i = 1,2, \dots j-1$

$\phi^*(e_j) = 4j-1$ where $e_j = u_j v_j$

$\phi^*(e_{j+i}) = 4j+2+4i$ where $e_{j+i} = u_{j+i} v_{j+i} \;\forall\; i = 1,2, \dots n-j-1$

$\phi^*(e_{n-1+i}) = 4i$ where $e_{n-1+i} = u_{i+1} v_i \;\forall\; i = 1,2, \dots j-1$

$\phi^*(e_{n-2+j+1}) = 4(j-1)+6$ where $e_{n-2+j+1} = u_{j+1} v_j$

$\phi^*(e_{n-2+j+1+i}) = 4(j-1)+6+4i$ where $e_{n-2+j+1+i} = u_{j+2} v_{j+1} \;\forall\; i = 1,2, \dots j-1$

$\phi^*(e_{2n-2+i}) = 5i$ where $e_{2n-2+i} = v_i v_{i+1} \;\forall\; i = 1,2, \dots n-2$

$\phi^*(e_j') = 4(j-1)+1$ where $e_j' = v_{j-1} w$,

$\phi^*(e_j'') = 4j$ where $e_j'' = w v_{j+1}$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_1) = 2,\; \phi^{**}(B_{k+1}) = 4k+1 \quad \forall\; k = 1,2,3, \dots j-2,\; \phi^{**}(B_j) = 4j-2,$

$\phi^{**}(B_{j+k}) = 4j-1+4k \quad \forall\; k = 1,2,3, \dots n-j-1,\; \phi^{**}(B_n) = 4(n-1)+2$

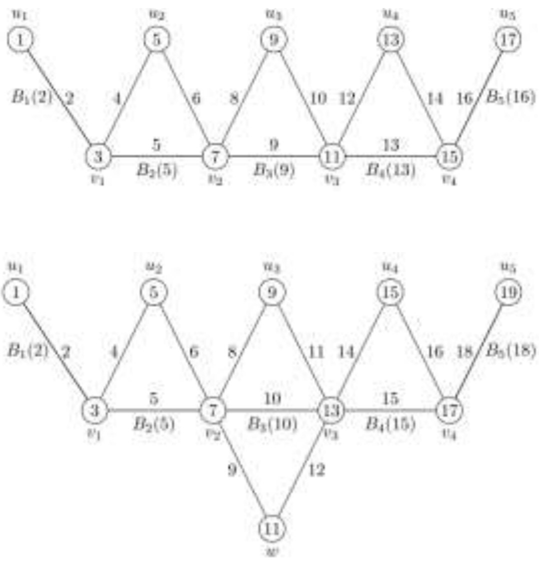


Figure 16: Middle graph of Path M(P_5) and AUM block mean labeling by duplication of a block B_3 in M(P_5) by a vertex w

In each case, ϕ^{**} is an injection and hence admits AUM block mean labeling.

Therefore, the graph $D_B(M(P_n))$ is an AUM block mean labelled graph.

Theorem 4.4: A Triangular snake graph T_n $(n \geq 2)$ is an AUM block mean labelled graph.

Proof: Consider G to be a Triangular graph T_n with the vertices $u_1, u_2, \ldots\ldots u_{2n-1}$, edges $e_1, e_2, \ldots\ldots e_{3n-3}$ and blocks $B_1, B_2, \ldots\ldots B_{n-1}$.

Define the bijective function $\phi: V(G) \to \{1,3,5, \ldots 2(2n-1)-1\}$ by

$\phi(u_i) = 2i - 1 \ \forall \ i = 1,2, \ldots 2n-1$, assigned as the label of the corresponding vertices.

Obtain the induced injective function $\phi^* : E(G) \to \{2,3,4, \ldots 2(2n-1)-2\}$ from ϕ by

$$\phi^*(e_i) = \frac{\phi(u_{2i-1}) + \phi(u_{2i+1})}{2} = 4i - 1 \quad \text{where } e_i = u_{2i-1}u_{2i+1} \ \forall \ i = 1,2, \ldots n-1$$

$$\phi^*\left(e_{j+n-1}\right) = \frac{\phi(u_j) + \phi(u_{j+1})}{2} = 2j \quad \text{where } e_{j+n-1} = u_j u_{j+1} \ \forall \ j = 1,2, \ldots 2n-2,$$

which are assigned as the labels of the edges of G.

From the labelled vertices and edges, the block labels are as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$$\phi^{**}(B_k) = \frac{\sum_{i=1}^{m} \phi(u_{k_i}) + \sum_{i=1}^{n} \phi^*(e_{k_i})}{No.of\ vertices\ incident\ with\ B_k + No.of\ edges\ incident\ with\ B_k}$$

$$= 4k - 1 \ \forall \ k = 1,2, \ldots n-1$$

For $k \neq j$, $\phi^{**}(B_k) \neq \phi^{**}(B_j)$, $1 \leq k, j \leq n-1$.

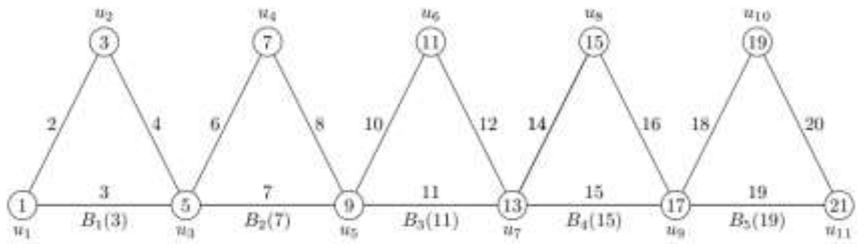


Figure 17: AUM block mean labeling of T_6

Hence, the Triangular snake graph T_n is an AUM block mean labelled graph.

Theorem 4.5: The graph $D_B(T_n), n \geq 2$ is an AUM block mean labelled graph.

Proof: Consider a Triangular snake graph T_n with the vertices $u_1, u_2, \ldots\ldots u_{2n-1}$, edges $e_1, e_2, \ldots\ldots e_{3n-3}$ and blocks $B_1, B_2, \ldots\ldots B_{n-1}$ and $D_{B_j}(T_n)$ be the graph produced through the duplication of a block B_j in T_n by a new vertex w.

Consider the following different cases for block B_j, when B_j is at the origin or at the terminus, and when B_j is an intermediate block of T_n.

Case 1: B_j is at the origin or at the terminus block of T_n.

Let us assume that B_j is at the origin block of T_n.

Therefore, $B_j = B_1$ and we get an additional block B_1' after duplicating the block B_1 by a vertex w.

Also, $D_{B_1}(T_n) = G$ has $2n$ vertices, $3n - 2$ edges and n blocks.

Define the bijective function $\phi : V(G) \to \{1,3,5, \dots 2(2n) - 1\}$ by

$\phi(u_1) = 1$, $\phi(u_2) = 5$, $\phi(u_3) = 9$, $\phi(u_4) = 7$, $\phi(w) = 3$

$\phi(u_{i+4}) = 9 + 2i \;\forall\; i = 1,2, \dots 2n - 5$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \to \{2,3,4, \dots 2(2n) - 2\}$ is defined from ϕ by

$\phi^*(e_1) = 5$ where $e_1 = u_1u_3$

$\phi^*(e_2) = 10$ where $e_2 = u_3u_5$,

$\phi^*(e_1') = 6$ where $e_1' = u_3w$,

$\phi^*(e_{2+i}) = 4i + 9$ where $e_{2+i} = u_{2i+3}u_{2i+5} \;\forall\; i = 1,2, \dots n - 3$

$\phi^*(e_n) = 3$ where $e_n = u_1u_2$

$\phi^*(e_{n+i}) = 6 + i$ where $e_{n+i} = u_{i+1}u_{i+2} \;\forall\; i = 1,2,3$

$\phi^*(e_{n+3+i}) = 10 + 2i$ where $e_{n+3+i} = u_{i+4}u_{i+5} \;\forall\; i = 1,2, \dots 2n - 6$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_1) = 5$, $\phi^{**}(B_1') = 6$, $\phi^{**}(B_2) = 9$

$\phi^{**}(B_{2+k}) = 4k + 9 \quad \forall\; k = 1,2, \dots n - 3$

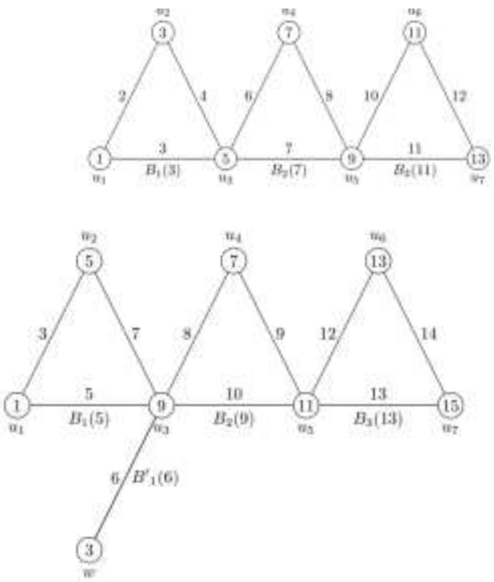


Figure 18: Triangular snake graph T_4 and AUM block mean labeling by duplication of a block B_1 in T_4 by a vertex w

Case 2:B_j $(2 \le j \le n-2)$ is an intermediate block of T_n

Then $D_{B_j}(T_n) = G$ has $2n$ vertices, $3n-1$ edges and $n-1$ blocks.

Define the bijective function $\phi : V(G) \to \{1,3,5,\dots 2(2n)-1\}$ by

$\phi(u_i) = 2i-1 \ \forall\ i = 1,2,\dots 2j, \quad \phi(w) = 4j+1$

$\phi(u_{2j+i}) = 4j+1+2i \ \ \forall\ i = 1,2,\dots 2n-2j-1$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \to \{2,3,4,\dots 2(2n)-2\}$ is defined from ϕ by

$\phi^*(e_i) = 4i-1$ where $e_i = u_{2i-1}u_{2i+1} \ \forall\ \ i = 1,2,\dots j-1$

$\phi^*(e_j) = 4j$ where $e_j = u_{2j-1}u_{2j+1}$

$\phi^*(e_j') = 4j-1$ where $e_j' = u_{2j-1}w$

$\phi^*(e_j'') = 4j+2$ where $e_j'' = wu_{2j+1}$

$\phi^*(e_{j+i}) = 4j+1+4i$ where $e_{j+i} = u_{2j+2i-1}u_{2j+2i+1} \ \forall\ \ i = 1,2,\dots n-j-1$

$\phi^*(e_{n-1+i}) = 2i$ where $e_{n-1+i} = u_iu_{i+1} \ \forall\ i = 1,2,\dots 2j-1,$

$\phi^*(e_{n+2j-1}) = 4j+1$ where $e_{n+2j-1} = u_{2j}u_{2j+1}$

$\phi^*(e_{n+2j+i-1}) = 4j+2+2i$ where $e_{n+2j+i-1} = u_{2j+i}u_{2j+i+1} \ \forall\ i = 1,2,\dots 2n-2j-2$

Now label the blocks as follows: Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_k) = 4k-1 \ \forall\ k = 1,2,\dots j-1$

$\phi^{**}(B_j) = 4j, \quad \phi^{**}(B_{j+k}) = 4i+13 \quad \forall\ k = 1,2,\dots n-j-1$

In each case, ϕ^{**} is an injection and hence admits AUM block mean labeling.

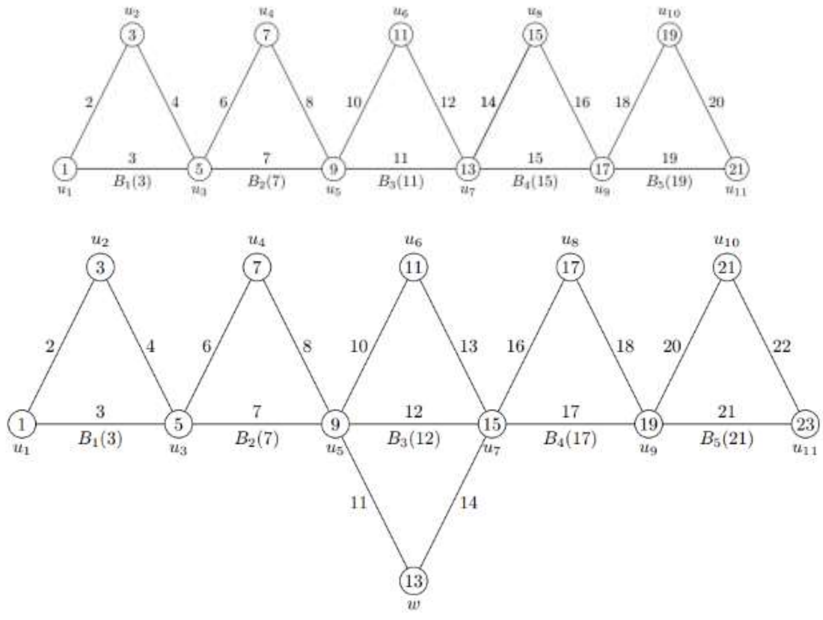


Figure 19: Triangular snake T_6 and AUM block mean labeling by duplication of a block B_3

in T_6 by a vertex w

Therefore, the graph $D_B(T_n)$ is an AUM block mean labelled graph.

Theorem 4.6: An Alternate triangular snake graph $A(T_n), n \geq 4$ is an AUM block mean labelled graph.

Proof: Consider G to be an Alternate triangular snake graph $A(T_n)$ with the vertices $u_1, u_2, \ldots\ldots u_{\lfloor\frac{3n}{2}\rfloor}$, edges $e_1, e_2, \ldots\ldots e_{2n-1}$ if n is even and $e_1, e_2, \ldots\ldots e_{2n-2}$ if n is odd and blocks $B_1, B_2, \ldots\ldots B_{n-1}$.

Define the bijective function $\phi : V(G) \to \left\{1,3,5, \ldots 2\lfloor\frac{3n}{2}\rfloor - 1\right\}$ by

$\phi(u_i) = 2i - 1 \;\; \forall\, i = 1,2, \ldots \lfloor\frac{3n}{2}\rfloor$, assigned as the label of the corresponding vertices.

Obtain the induced injective function $\phi^* : E(G) \to \left\{2,3,4, \ldots 2\lfloor\frac{3n}{2}\rfloor - 2\right\}$ from ϕ by

$\phi^*(e_i) = 6i - 3$ where $e_i = u_{3i-2}u_{3i} \;\forall\, i = 1,2, \ldots \lfloor\frac{n}{2}\rfloor$

$\phi^*\left(e_{j+\lfloor\frac{n}{2}\rfloor}\right) = 2j$ where $e_{j+\lfloor\frac{n}{2}\rfloor} = u_j u_{j+1} \;\forall\, j = 1,2, \ldots \left\lfloor\frac{3n}{2}\right\rfloor - 1$,

which are assigned as the labels of the edges of G.

From the labelled vertices and edges, the block labels are as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_k) = 3k \;\; \forall\, k = 1,2, \ldots n - 1$

For $k \neq j,\; \phi^{**}(B_k) \neq \phi^{**}(B_j)$, $1 \leq k, j \leq n - 1$.

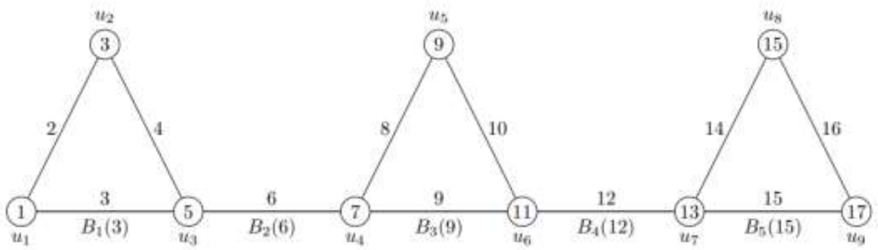


Figure 20: AUM block mean labeling of $A(T_6)$

Hence, the Alternate triangular snake graph $A(T_n)$ is an AUM block mean labelled graph.

Theorem 4.7: The graph $D_B(A(T_n)),\; n \geq 4$ is an AUM block mean labelled graph.

Proof: Consider an Alternate triangular snake graph $A(T_n)$ with the vertices $u_1, u_2, \ldots\ldots u_{\lfloor\frac{3n}{2}\rfloor}$, $e_1, e_2, \ldots\ldots e_{2n-1}$ if n is even and $e_1, e_2, \ldots\ldots e_{2n-2}$ if n is odd and blocks $B_1, B_2, \ldots\ldots B_{n-1}$ and $D_{B_j}(A(T_n))$ be the graph produced through the duplication of a block B_j in $A(T_n)$ by a new vertex w.

Case 1: n is even

Consider the following different cases for block B_j, when B_j is at the origin or at the terminus, and when B_j is an intermediate block of $A(T_n)$.

Sub Case 1: B_j is at the origin or at the terminus block of $A(T_n)$.

Let us assume that B_j is at the origin block of $A(T_n)$.

Therefore, $B_j = B_1$ and we get an additional block B_1' after duplicating the block B_1 by a vertex w.

Also, $D_{B_1}(A(T_n)) = G$ has $\frac{3n}{2} + 1$ vertices, $2n$ edges and n blocks.

Define the bijective function $\phi : V(G) \rightarrow \left\{1,3,5, \dots 2(\frac{3n}{2} + 1) - 1\right\}$ by

$\phi(u_i) = 2i - 1, \quad \forall\ i = 1,2,3,\ \phi(w) = 7$

$\phi(u_{3+i}) = 7 + 2i, \ \forall\ i = 1,2, \dots \frac{3n}{2} - 3$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \rightarrow \left\{2,3,4, \dots 2\left(\frac{3n}{2} + 1\right) - 2\right\}$ is defined from

ϕ by $\phi^*(e_1) = 3$ where $e_1 = u_1 u_3$,

$\phi^*(e_1') = 6$ where $e_1' = u_3 w$

$\phi^*(e_{1+i}) = 6i + 5$ where $e_{1+i} = u_{3i+1} u_{3i+3}\ \forall\ i = 1,2, \dots \frac{n}{2} - 1$

$\phi^*\left(e_{\frac{n}{2}+i}\right) = 2i$ where $e_{\frac{n}{2}+i} = u_i u_{i+1}\ \forall\ i = 1,2$

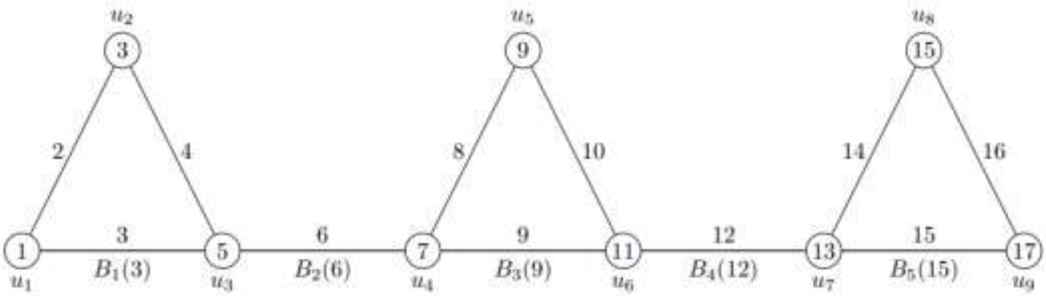
$\phi^*\left(e_{\frac{n}{2}+3}\right) = 7$ where $e_{\frac{n}{2}+3} = u_3 u_4$

$\phi^*\left(e_{\frac{n}{2}+3+i}\right) = 8 + 2i$ where $e_{\frac{n}{2}+3+i} = u_{3+i} u_{3+i+1}\ \forall\ i = 1,2, \dots \frac{3n}{2} - 4$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \rightarrow Z^+$ by

$\phi^{**}(B_1) = 3,\ \phi^{**}(B_1') = 6,\ \phi^{**}(B_2) = 7,\ \phi^{**}(B_{2+k}) = 3k + 8\ \forall\ k = 1,2, \dots n - 3$



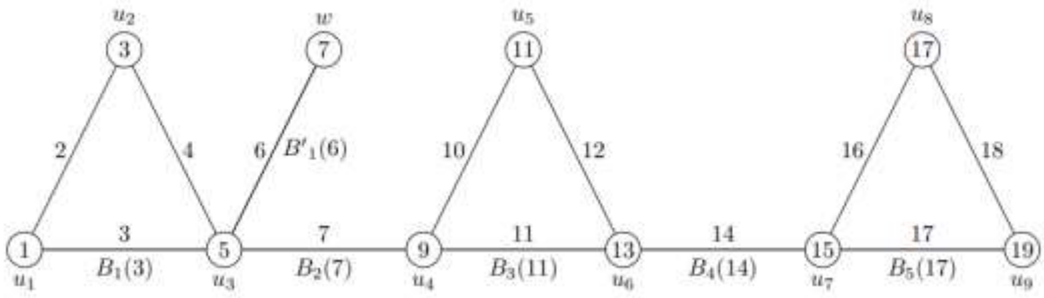


Figure 21: Alternate Triangular snake $A(T_6)$ and AUM block mean labeling by duplication of a block in B_1 $A(T_6)$ by a vertex w

Sub Case 2: B_j $(2 \leq j \leq n-2)$ is at the intermediate block of $A(T_n)$

Therefore, B_j is an edge or C_3.

(*i*) B_j is an edge

Then $D_{B_j}(A(T_n)) = G$ has $\frac{3n}{2}+1$ vertices, $2n+1$ edges and $n-1$ blocks.

Define the bijective function $\phi : V(G) \to \left\{1,3,5,\ldots 2(\frac{3n}{2}+1) - 1\right\}$ by

$\phi(u_i) = 2i-1 \quad \forall\, i = 1,2,\ldots 3\left(\frac{j}{2}\right),\quad \phi(w) = 3j+1,$

$\phi\left(u_{3\left(\frac{j}{2}\right)+i}\right) = 3j+1+2i \quad \forall\, i = 1,2,\ldots \frac{3n}{2} - \frac{3j}{2}$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \to \left\{2,3,4,\ldots 2\left(\frac{3n}{2}+1\right) - 2\right\}$ is defined from

ϕ by $\phi^*(e_i) = 6i-3$ where $e_i = u_{3i-2}u_{3i}\ \forall\, i = 1,2,\ldots \frac{j}{2}$

$\phi^*\left(e_{\frac{j}{2}+i}\right) = 3j-1+6i$ where $e_{\frac{j}{2}+i} = u_{3\frac{j}{2}+3i-2}u_{3\frac{j}{2}+3i}\ \forall\, i = 1,2,\ldots \frac{n}{2} - \frac{j}{2}$

$\phi^*(e'_j) = 3j$ where $e'_j = u_{\frac{3j}{2}}w,$

$\phi^*(e''_j) = 3j+2$ where $e''_j = u_{\frac{3j}{2}+1}w$

$\phi^*\left(e_{\frac{n}{2}-\frac{j}{2}+i}\right) = 2i$ where $e_{\frac{n}{2}-\frac{j}{2}+i} = u_iu_{i+1}\ \forall\, i = 1,2,\ldots \frac{3j}{2} - 1$

$\phi^*\left(e_{\frac{n}{2}+j}\right) = 3j+1$ where $e_{\frac{n}{2}+j} = u_{\frac{3j}{2}}u_{\frac{3j}{2}+1}$

$\phi^*\left(e_{\frac{n}{2}+j+i}\right) = 3j+2+2i$ where $e_{\frac{n}{2}+j+i} = u_{\frac{3j}{2}+i}u_{\frac{3j}{2}+i+1}\ \forall\, i = 1,2,\ldots \frac{3n}{2} - \frac{3j}{2} - 1$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_k) = 3k\ \forall\, k = 1,2,\ldots j-1$

$\phi^{**}(B_j) = 3j+1,\ \phi^{**}(B_{j+k}) = 3j+2+3k\ \ \forall\, k = 1,2,\dots n-j-1$

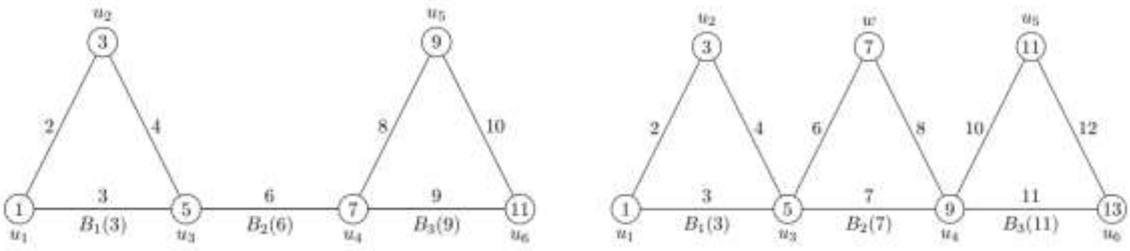


Figure 22: Alternate Triangular snake $A(T_4)$ and AUM block mean labeling by duplication of a block B_2 in $A(T_4)$ by a vertex w

(ii) B_j is C_3

Then $D_{B_j}(A(T_n)) = G$ has $\frac{3n}{2}+1$ vertices, $2n+1$ edges and $n-1$ blocks.

Define the bijective function $\phi: V(G) \to \left\{1,3,5,\dots 2(\frac{3n}{2}+1) - 1\right\}$ by

$\phi(u_i) = 2i-1\ \ \forall\, i = 1,2,\dots \frac{3(j-1)}{2}+2,\ \ \phi(w) = 3j+2,$

$\phi\left(u_{\frac{3(j-1)}{2}+2+i}\right) = 3j+2+2i\ \ \forall\, i = 1,2,\dots \frac{3n-3j-1}{2}$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^*: E(G) \to \left\{2,3,4,\dots 2\left(\frac{3n}{2}+1\right) - 2\right\}$ is defined from

ϕ by $\phi^*(e_i) = 6i-3$ where $e_i = u_{3i-2}u_{3i}\ \forall\, i = 1,2,\dots \lfloor \frac{j}{2} \rfloor$

$\phi^*\left(e_{\lfloor \frac{j}{2} \rfloor+1}\right) = 6\left\lfloor \frac{j}{2} \right\rfloor + 4$ where $e_{\lfloor \frac{j}{2} \rfloor+1} = u_{3\lfloor \frac{j}{2} \rfloor+1}u_{3\lfloor \frac{j}{2} \rfloor+3}$

$\phi^*\left(e_{\lfloor \frac{j}{2} \rfloor+1+i}\right) = 6\left\lfloor \frac{j}{2} \right\rfloor + 5 + 6i$ where $e_{\lfloor \frac{j}{2} \rfloor+1+i} = u_{3\lfloor \frac{j}{2} \rfloor+3i+1}u_{3\lfloor \frac{j}{2} \rfloor+3i+3}\ \forall\, i = 1,2\dots \frac{n}{2} - \left\lfloor \frac{j}{2} \right\rfloor - 1$

$\phi^*\left(e_{\frac{n}{2}+i}\right) = 2i$ where $e_{\frac{n}{2}+i} = u_iu_{i+1}\ \forall\, i = 1,2\dots \left\lfloor \frac{3j}{2} \right\rfloor$

$\phi^*\left(e_{\frac{n}{2}+\lfloor \frac{3j}{2} \rfloor+1}\right) = 2\left\lfloor \frac{3j}{2} \right\rfloor + 3$ where $e_{\frac{n}{2}+\lfloor \frac{3j}{2} \rfloor+1} = u_{\lfloor \frac{3j}{2} \rfloor+1}u_{\lfloor \frac{3j}{2} \rfloor+2}$

$\phi^*\left(e_{\frac{n}{2}+\lfloor \frac{3j}{2} \rfloor+1+i}\right) = 2\left\lfloor \frac{3j}{2} \right\rfloor + 4 + 2i$ where $e_{\frac{n}{2}+\lfloor \frac{3j}{2} \rfloor+1+i} = u_{\lfloor \frac{3j}{2} \rfloor+1+i}u_{\lfloor \frac{3j}{2} \rfloor+1+i+1}$

$\forall\, i = 1,2\dots \frac{3n}{2} - \left\lfloor \frac{3j}{2} \right\rfloor - 2$

$\phi^*(e_j') = 2\left\lfloor \frac{3j}{2} \right\rfloor + 1$ where $e_j' = u_{\lfloor \frac{3j}{2} \rfloor}w,$

$\phi^*(e_j'') = 2\left\lfloor \frac{3j}{2} \right\rfloor + 4$ where $e_j'' = u_{\lfloor \frac{3j}{2} \rfloor+2}w$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}\ (B_k) = 3k\ \forall\ k = 1,2, \dots j-1$

$\phi^{**}\ (B_j) = 3j+1,\ \phi^{**}\ (B_{j+k}) = 3j+2+3k\ \ \forall\ k = 1,2, \dots n-j-1$

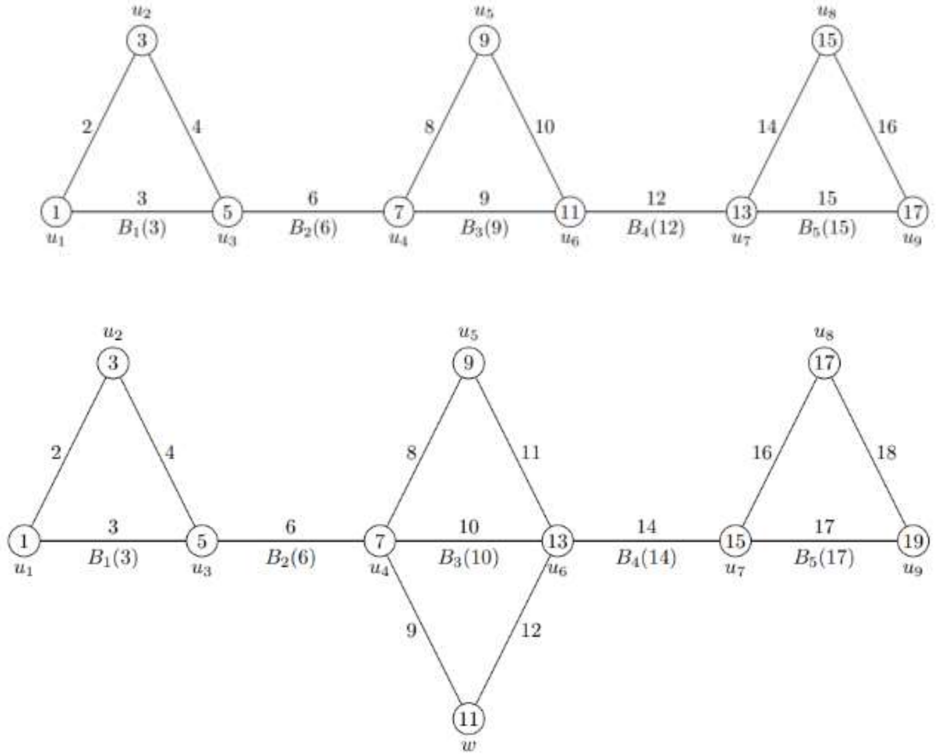


Figure 23: Alternate Triangular snake $A(T_6)$ and AUM block mean labeling by duplication of a block B_3 in $A(T_6)$ by a vertex w

Case 2: n is odd

Consider the following different cases for block B_j, when B_j is at the origin or at the terminus, and when B_j is an intermediate block of $A(T_n)$.

Sub Case 1: B_j is at the origin block of $A(T_n)$

Therefore, $B_j = B_1$ and we get an additional block B_1'.

Also, $D_{B_1}(A(T_n)) = G$ has $\left\lfloor\frac{3n}{2}\right\rfloor + 1$ vertices, $2n-1$ edges and n blocks.

By replacing $\frac{3n}{2}$ by $\lfloor\frac{3n}{2}\rfloor$ and $\frac{n}{2}$ by $\lfloor\frac{n}{2}\rfloor$ in the Sub case 1 of Case 1, AUM block mean labeling of blocks is obtained.

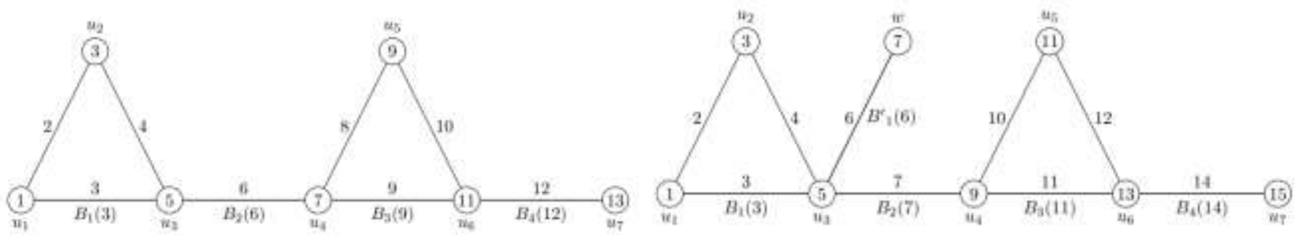


Figure 24: Alternate Triangular snake $A(T_5)$ and AUM block mean labeling by duplication of a block B_1 in $A(T_5)$ by a vertex w

Sub Case 2: B_j is at the terminus block of $\boldsymbol{A}(T_n)$

Therefore, $B_j = B_{n-1}$ and we get an additional block B'_{n-1}.

Also, $D_{B_{n-1}}(A(T_n)) = G$ has $\left\lfloor\frac{3n}{2}\right\rfloor + 1$ vertices, $2n-1$ edges and n blocks.

Define the bijective function $\phi: V(G) \to \left\{1,3,5, \ldots 2(\lfloor\frac{3n}{2}\rfloor + 1) - 1\right\}$ by

$\phi(u_i) = 2i - 1 \;\forall\; i = 1,2, \ldots \lfloor\frac{3n}{2}\rfloor$

$\phi(w) = 2\lfloor\frac{3n}{2}\rfloor + 1$, assigned as the label of the corresponding vertices.

Here, the induced injective function $\phi^* : E(G) \to \left\{2,3,4, \ldots 2\left(\left\lfloor\frac{3n}{2}\right\rfloor + 1\right) - 2\right\}$ is defined from ϕ by $\phi^*(e_i) = 6i - 3$ where $e_i = u_{3i-2}u_{3i} \;\forall\; i = 1,2, \ldots \lfloor\frac{n}{2}\rfloor$

$\phi^*\left(e_{\left\lfloor\frac{n}{2}\right\rfloor + i}\right) = 2i$ where $e_{\left\lfloor\frac{n}{2}\right\rfloor+i} = u_i u_{i+1} \;\forall\; i = 1,2, \ldots \left\lfloor\frac{3n}{2}\right\rfloor - 1$

$\phi^*(e'_{n-1}) = 3n - 2$ where $e'_{n-1} = u_{n-1}w$

Now label the blocks as follows:

Define $\phi^{**}: B(G) \to Z^+$ by

$\phi^{**}(B_k) = 3k \;\forall\; k = 1,2, \ldots n-1$

$\phi^{**}(B'_{n-1}) = 3n - 2$

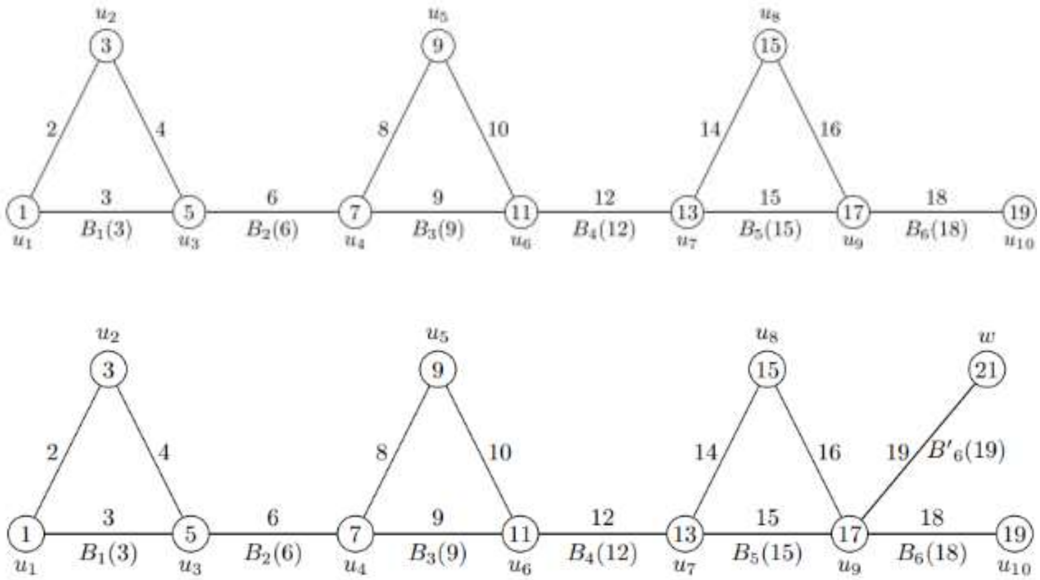


Figure 25: Alternate triangular snake $A(T_7)$ and AUM block mean labeling by duplication of a block B_6 in $A(T_7)$by a vertex w

Sub Case 3: $B_j(2 \leq j \leq n-2)$ is at the intermediate block of $\boldsymbol{A}(T_n)$

Therefore, B_j is an edge or C_3.

(i) B_j is an edge

Then $D_{B_j}(A(T_n)) = G$ has $\left\lfloor \frac{3n}{2} \right\rfloor + 1$ vertices, $2n$ edges and $n-1$ blocks.

By replacing $\frac{3n}{2}$ by $\lfloor \frac{3n}{2} \rfloor$ and $\frac{n}{2}$ by $\lfloor \frac{n}{2} \rfloor$ in the Sub case 2 (B_j is an edge) of Case 1, AUM block mean labeling of blocks is established.

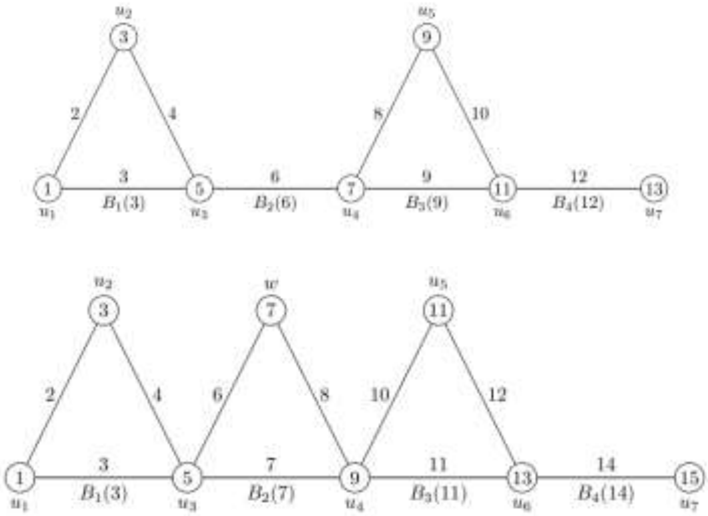
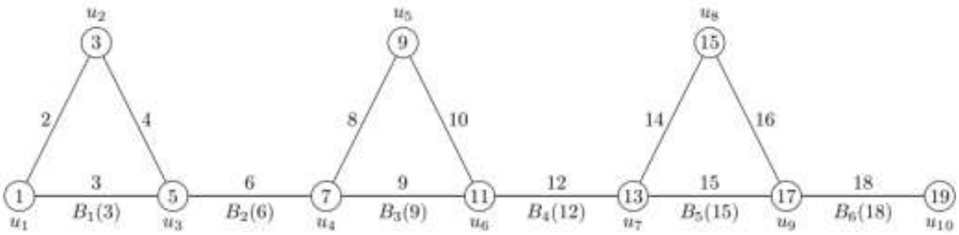


Figure 26: Alternate triangular snake $A(T_5)$ and AUM block mean labeling by duplication of a block B_2 in $A(T_5)$ by a vertex w

(ii) B_j is C_3

Then $D_{B_j}(A(T_n)) = G$ has $\left\lfloor \frac{3n}{2} \right\rfloor + 1$ vertices, $2n$ edges and $n-1$ blocks.

By replacing $\frac{3n}{2}$ by $\lfloor \frac{3n}{2} \rfloor$ and $\frac{n}{2}$ by $\lfloor \frac{n}{2} \rfloor$ in the Sub case 2 (B_j is C_3) of Case 1, AUM block mean labeling of blocks is established.



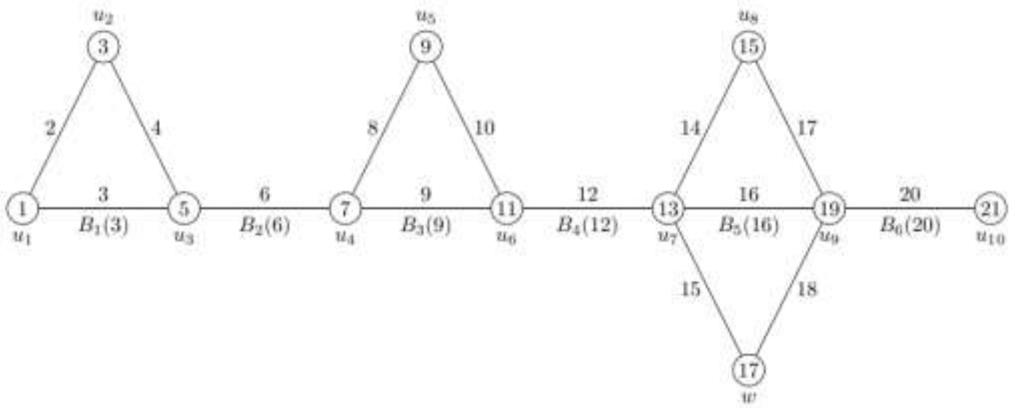


Figure 27: Alternate triangular snake $A(T_7)$ and AUM block mean labeling by duplication of a block B_5 in $A(T_7)$ by a vertex w

In each case, ϕ^{**} is an injection and hence admits AUM block mean labeling.

Therefore, the graph $D_B(A(T_n))$ is an AUM block mean labelled graph.

5. Application of duplication of a block by a vertex in Water Management

The notion of duplication of blocks in graphs can be applied in the drainage system and water supply to relieve the blockage in the system so as to enhance the efficiency and reliability of water management. By duplicating drainage blocks, the system can maintain functionality even if one block fails. The following algorithm gives the enhanced way for water management:

Algorithm:

1. Represent the drainage system as the Path graph $P_n(n \geq 3)$[Here, vertices and edges denote the fitting point and the water flow pipe respectively].
2. Consider a blockage in the block B_i of P_n.
3. By applying the concept of duplication of blocks by a vertex, the block B_i is duplicated by a vertex w. Hence, we obtain a graph $D_{B_i}(P_n)$ with the new pathway, and the flow of water can be continued.
4. If there is any blockage in the block B_j in $D_{B_i}(P_n)$, repeat step 3.
5. Repeat the same procedure until all the blockages are duplicated and ensure efficient water flow.

Illustration:

Consider a Path graph P_5, suppose there is a blockage or some repair in the block B_3, the graph obtained after duplication of the block B_3 by a vertex w, help us to overcome the problem. By joining v_3 to w and w to v_4, the new pathway v_3wv_4 will be available. Figure 28 shows the alternate pathway to handle the such situation.

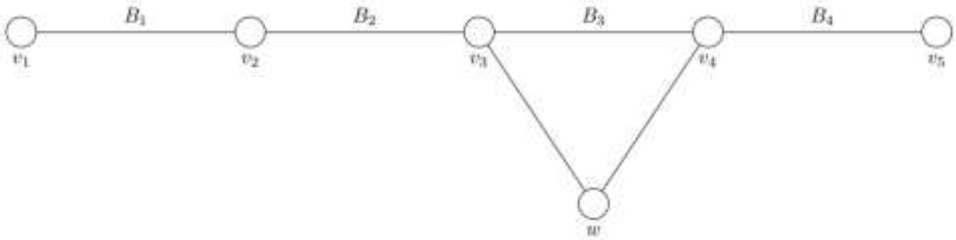


Figure 28: Graph $D_{B_3}(P_5)$

Suppose there is a block in B_2 and B_3, then Figure 29 represents the alternate pathway by joining w to v_3 and v_4.

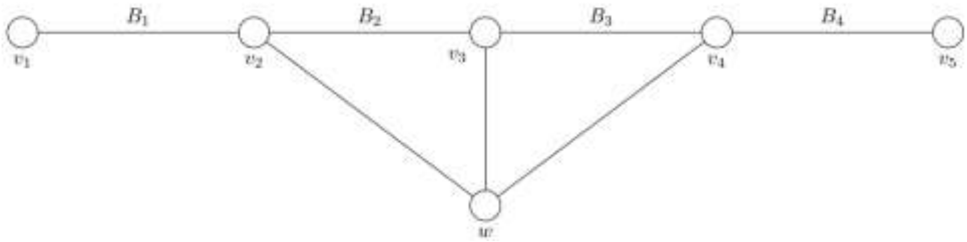


Figure 29: Graph $D^2_{B_2,B_3}(P_5)$

This will be helpful for preventing flooding and managing stormwater effectively. Duplicating drainage systems can accommodate larger volumes of water, especially in areas prone to heavy rainfall or runoff. With duplicated blocks, maintenance can be done without disrupting the overall drainage function, ensuring continuous water flow management.

Conclusion

In the present work, the new concept of duplication of a block by a vertex and duplication of multiple blocks by a vertex in a graph is introduced and discussed for Path graph. The concept of duplication of a block by a vertex can be applied in real-life situations, such as water supply, drainage connection, cable connection, etc., and it was established that the graphs constructed through the duplication of a block by a vertex admit AUM block mean labeling for Path, Middle graph of Path, Triangular snake, and Alternate triangular snake graphs. This study provides an application in the water management through an algorithm. There is a further scope for extension of AUM block mean labeling for the graphs constructed through duplication of a block by a vertex for standard graphs.

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