Application of Leslie Matrix Model in Human Population Dynamics and Public Health

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ABSTRACT

This paper delves into the Leslie Matrix Population Model to examine age specific density dependence in human population dynamics. The model's key properties and the implications of its eigen values are explored in details, providing a foundational understanding of its mathematical underpinnings. A significant portion of the study is dedicated to the practical application of the Leslie matrix, wherein we project India's age-specific female numbers between 2011 and 2051 at five-years intervals. By constructing a Leslie matrix tailored to India's demographic data, we are able to predict population changes with a focus on female age groups. To demonstrate the model's effectiveness and validate its accuracy, we compared the projected population for the year 2016 against actual data provided by the Registrar General of India, Govt. of India. The comparison indicates a high degree of accuracy, underscoring the model's robustness in demographic predictions. This study not only confirm the utility of Leslie Matrix population model in forecasting population dynamics but also highlight its potential for planning and policy making in the context of India's demographic landscape. Through rigorous analysis and validation, we affirm the model's capacity to provide valuable insight into future population trends based on current age specific data.

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Introduction

Matrix population models have become essential tools in population mathematics, particularly for studying human population dynamics and predicting population growth. In 1945, P.H. Leslie introduced a population projection matrix, now known as the Leslie Matrix Population Model. He later refined this model in 1959 to account for the effects of other population members on population growth and structure. Leslie matrix are crucial in understanding how populations are influenced by age specific survival and fertility rates this age dependent matrix model is widely applied in human population studies to predict population growth and age distribution over time(1,2). The Leslie Model is particularly effective for determining population growth using fertility rates, survival rates and base population data. Matrix population model offer valuable insight into the relationship between demographic process and population dynamics. Ecologists use it to assess species survival in new environment conditions and predict long term population trends(3). It is a discrete matrix model and, a crucial tool in population dynamics to examine the impact of age specific survivorship and fertility rates on population growth and age distribution(4).The Leslie model is a discrete, age structured model that uses a matrix to describe the transitions of individuals between age classes in a population(5). Recent studies have focused on its eigen values and eigen vectors, which are critical for understanding long term population behaviour including growth rates, stable age structure and distribution. A notable property of the Leslie Matrix is the dominant eigen value which represents asymptotic population growth, while corresponding eigen vector gives the stable age distribution and these properties are pivotal for predicting demographic changes and formulating population policies (6,7). A study by (7) utilized the Leslie matrix to project the age specific population of united states, highlighting its utility in anticipating shifts in age structure and dependency ratios. The study demonstrated how incorporating fertility and mortality rates into matrix model provides reliable projection that are essential for policy planning(8,9). A matrix model applied on population to evaluate China's two child policy, showing how changes in fertility rates influenced future population structure and findings underscore the model's effectiveness in simulation the outcomes of policy interventions on population dynamics(10). Some advancements have included perturbation analysis to explore how demographic process can modify population trajectories, incorporating new theoretical development in deterministic transient analysis(11–13). A perturbative formalism for linear non negative matrix difference had been included in matrix model. This framework defines the effective growth rate as the non-trivial effective eigen value, which characterizes the long-term asymptotic dynamics of mean value population vector state. Through random Leslie analysis, researchers demonstrated that incorporating random vital parameters results in an asymptotic mean value vector growth rate lower than previously estimated(14–18). This research paper aims to investigate the impacts of age-specific density dependence on population growth and structure. By employing a density dependent Leslie matrix model, the study seeks to analyse how varying birth and survival rates across different age groups influence over all population dynamics. The objective is to enhance the accuracy of population projection and provide deeper insights into demographic patterns.

Model Formulation

We consider that, the population is divided into m distinct age groups $y_1(t), y_2(t), ...y_m(t)$ for given $y_1(0), y_2(0), ...y_m(0)$. The population of these age groups at any given time t are determined, assuming constant birth and death rates within each group. The model evaluates population changes in discrete stages, assessing the population at time t and $t+1$. Due to this approach, it is referred to as a discrete-time discrete-age-scale population model. This is a single-sex model focusing solely on female population changes, assuming proportional changes in the male population, which are considered consistent. Let $f_a \geq 0$ (for a=1, 2, 3…m) represent the number of females born, on average to a female of the age a. The female population in age group $(a-1, a)$ at time t is $y_a(t)$, each of whom gives to a certain number of females in the interval $(t, t+1)$. Therefore, the numbers of females in the age-group (0,1) alive at time $t+1$ of these $y_a(t)$ females are $f_a y_a(t)$.Thus, the number of females in group (0, 1) at the time $t+1$ is,

$$y_1(t+1) = f_1 y_1(t) + f_2 y_2(t) + \cdots + f_m y_m(t) \tag{1}$$

Let $P_a(a=1,2,3,......,m-1)$ be the probability of population of a^{th} age-group at time t, who alive to population of $(a+1)^{th}$ age-group at the time $t+1$. Therefore, we have,

$$y_{a+1}(t+1)=P_a y_a(t) \text{ where } a=1,2,3,......m \tag{2}$$

From equation (2), we have

$$\left.\begin{array}{l} y_2(t+1)=P_1 y_1(t) \\ y_3(t+1)=P_2 y_2(t) \\ y_4(t+1)=P_3 y_3(t) \\ \vdots \\ y_m(t+1)=P_{m-1} y_{m-1}(t) \end{array}\right\} \tag{3}$$

The matrix form of system of linear difference equation (1), (2) and (3) is given by

$$\begin{pmatrix} y_1(t+1) \\ y_2(t+1) \\ y_3(t+1) \\ . \\ . \\ . \\ y_m(t+1) \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \dots & f_{m-1} & f_m \\ P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ 0 & 0 & \dots & P_{m-1} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ . \\ . \\ . \\ y_m(t) \end{pmatrix} \tag{4}$$

Or
$$Y(t+1)=LY(t) \tag{5}$$

$$L=\begin{pmatrix} f_1 & f_2 & \dots & f_{m-1} & f_m \\ P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ 0 & 0 & \dots & P_{m-1} & 0 \end{pmatrix} \tag{6}$$

$$Y(t)=[y_1(t)\ y_2(t).......y_m(t)]^T \tag{7}$$

The matrix L is known as Leslie matrix. In Leslie matrix all the elements are non-negative i.e., $f_a \geq 0$ and $0 \leq f_a < 0 \quad \forall$ all a's.

Equation (5) is a recurrence relation, so put $t-1, t-2, t-3, ...$ in place of t in (5) we get

$$Y(t)=LY(t-1),\ Y(t-1)=LY(t-2),\ Y(t-2)=LY(t-3) \text{ and so on.}$$

From above obtained linear difference equation, we get

$$Y(t)=L^2Y(t-2)=L^3Y(t-3)=\cdots\cdots=L^tY(0) \tag{8}$$

From equation (8), we can find $y_1(t), y_2(t), ...y_m(t)$ for any positive integer t. If the Leslie matrix L has distinct Eigen values, say $\lambda_1, \lambda_2, \lambda_3, ...\lambda_m$, then

$$L = X \text{ Dig.}[\lambda_1, \lambda_2, \lambda_3, ...\lambda_m] X^{-1} = XDX^{-1} \tag{9}$$

Where X is the matrix whose columns are Eigen vectors corresponding to $\lambda_1, \lambda_2, \lambda_3, ...\lambda_m$ and X^{-1} is it's inverse and D is the diagonal matrix with its diagonal elements as $\lambda_1, \lambda_2, \lambda_3, ...\lambda_m$.

Now,

$$L^2 = (XDX^{-1})^2 = (XDX^{-1})(XDX^{-1}) = XD(X^{-1})DX^{-1} = XD^2X^{-1}$$

$$L^t = XD^tX^{-1} \tag{10}$$

From equation (8) and (10), we get

$$Y(t) = XD^tX^{-1}Y(0) \tag{11}$$

On simplifying (11), we thus get the a^{th} solution,

$$Y_a(t) = b_{a1}\lambda_1{}^t + b_{a2}\lambda_2{}^t + ... + b_{am}\lambda_m{}^t \tag{12}$$

For $a = 1, 2, 3, ...m.$

Interpretation of Model

The asymptotic behaviour of $y_a(t)$ will depend on the largest absolute Eigen value. Let his absolute value be λ_0. Then we the following conclusions:

(i) If $\lambda_o > 1$, $y_a(t) \to \infty$, as $t \to \infty$, which shows that all age-group's population will increase.

(ii) If $\lambda_o < 1$, $y_a(t) \to 0$, as $t \to \infty$, implying that all the populations will go to extinction.

(iii) If $\lambda_o = 1$, $y_0(t) \to b_{a1} + b_{a2} + b_{a3} + ... + b_{am}$ (a constant) as $t \to \infty$ showing that asymptotically stationary populations exist across all age groups.

Dominant Eigen Value of Leslie Matrix

Based on the aforementioned conclusion, it is evident that the eigen values of matrix L, specially the dominant eigen value of L, are significant. For this reason, we will address the eigen values and corresponding eigen vectors of L in this section.

Since the Leslie matrix L is non-negative and for some positive integers m, the elements of L^m are all positive if any two consecutive fecundity measures $f_{1,} f_{2,} f_{3,} ....f_m$ are non-zero.

Thus, we have the following theorem for Leslie matrix:

Theorem: Let L be a Leslie matrix with at least one pair of consecutive non zero fecundity measures ($f_m \neq 0$). Then L has a positive eigenvalue of algebraic multiplicity one. This eigenvalue is greater in absolute value than any other eigenvalue of L, and it correspond to an eigenvector with only positive elements.

Proof: Leslie matrix L is given by

$$L = \begin{pmatrix} f_1 & f_2 & \cdots & f_{m-1} & f_m \\ P_1 & 0 & \ldots & 0 & 0 \\ 0 & P_2 & \ldots & 0 & 0 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ 0 & 0 & \ldots & P_{m-1} & 0 \end{pmatrix}_{m\times m}$$

And characteristic equation given by

$$\phi(\lambda) = |L - \lambda I| = 0 \tag{13}$$

Where I is a unit matrix of order $m \times m$. Equation (13) may be rewritten as

$$\phi(\lambda) = \begin{vmatrix} f_1\text{-}\lambda & f_2 & \cdots & f_{m-1} & f_m \\ P_1 & \text{-}\lambda & \ldots & 0 & 0 \\ 0 & P_2 & \ldots & 0 & 0 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ 0 & 0 & \ldots & P_{m-1} & \text{-}\lambda \end{vmatrix} = 0$$

$$\phi(\lambda) = \lambda^m - f_1\lambda^{m-1} - f_2P_1\lambda^{m-2} - f_3P_1P_2\lambda^{m-3} \ldots\ldots\ldots - f_mP_1P_2 \ldots\ldots\ldots P_{m-1} = 0 \tag{14}$$

Since $P_1, P_2, P_3, \ldots\ldots P_{m-1}$ are all positive and it is assumed that $f_n \neq 0$ i.e., $f_n > 0$; then we have

$$\phi(0) = -f_mP_1P_2 \ldots\ldots\ldots P_{m-1} < 0 \text{ and } \phi(\infty) = \infty > 0$$

Now $\phi(0) < 0$ and $\phi(\infty) > 0$ imply that the equation $\phi(\lambda) = 0$ has a minimum of one positive real root.

Obliviously $\phi(0) \neq 0$, so that 0 is not the eigen value of $\phi(\lambda) = 0$ .therefore divide (14) by λ^m, we get

$$F(\lambda) = f_1\lambda^{-1} + P_1f_2\lambda^{-2} + \cdots + f_mP_1P_2\cdots P_{m-1}\lambda^{m-1} = 1 \tag{15}$$

Differentiating both sides of (15) w.r.t. λ, we get

$$\frac{dF}{d\lambda} = -\frac{f_1}{\lambda^2} - \frac{2P_1f_2}{\lambda^3} - \cdots - \frac{mP_1P_2\cdots P_{m-1}f_m}{\lambda^{m+1}} \tag{16}$$

Obviously, $\frac{dF}{d\lambda} < 0$ when $\lambda > 0$ which implies that $F(\lambda)$ falls monotonically as it rises from 0 to infinity as it does so. Consequently, there is just one real positive root of $F(\lambda) = 1$ with multiplicity one, let it be λ_0

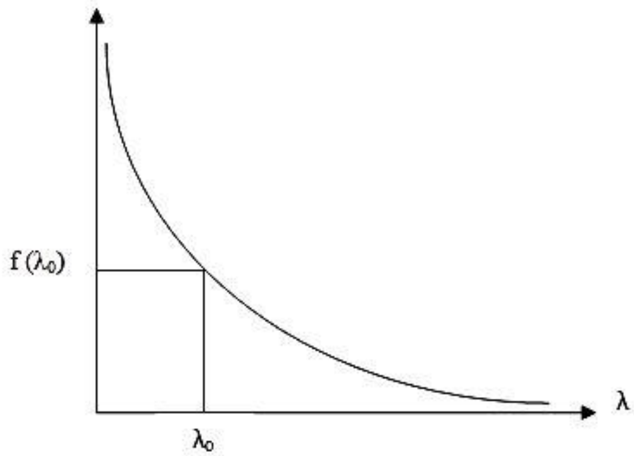

Fig.1: Graph of F(λ)

Let $\lambda_j^{-1} = e^{\alpha+i\beta}$ where λ_j be any other eigen value, with α and β are real and positive and $\beta \neq 2k\pi$, so that

$$\lambda_j = \frac{1}{e^{\alpha+i\beta}} = \frac{1}{e^{\alpha}} e^{-i\beta} = e^{-(\alpha+i\beta)}$$

Therefore, λ_j is either negative real or complex.

Now putting $\lambda_j^{-1} = e^{\alpha+i\beta}$ in (15), we get

$$f_1 e^{\alpha+i\beta} + P_1 f_2 e^{2\alpha+2i\beta} + ... + f_m P_1 P_2 ....P_{m-1} e^{m\alpha+im\beta} = 1$$

Or by Equating the real parts, we get

$$f_1 e^{\alpha} \cos\beta + P_1 f_2 e^{2\alpha} \cos 2\beta + ... + f_m P_1 P_2 ....P_{m-1} e^{m\alpha} \cos m\beta = 1 \qquad (17)$$

Let f_i and f_{i+1} be both non-zero and, $\cos(i\beta)$ and $\cos((i+i)\beta)$ cannot be unity because $\beta \neq 2k\pi$ from (17) it is clear that $\cos i\beta < 1$and $\cos((i+i)\beta) < 1$.

Now comparing (17) with the equation

$$f_1 \lambda_0^{-1} + P_1 f_2 \lambda_0^{-2} + ... - f_m P_1 P_2 ....P_{m-1} \lambda_0^{m-1} = 1 \qquad (18)$$

We get, $e^{\alpha} \cos\beta = \lambda_0^{-1}$ and $\cos\beta < 1$ as $\beta \neq 2k\pi$

So that $e^{\alpha} \cos\beta < e^{\alpha} \Rightarrow \lambda_0^{-1} < e^{\alpha}$

Also $\lambda_0^{-1} = e^{\alpha+i\beta}$ so that $\left|\lambda_j^{-1}\right| = e^{\alpha}$ and $\lambda_0^{-1} < \left|\lambda_j^{-1}\right| \Rightarrow \left|\lambda_j\right| < \lambda_0$

Thus, λ_0 is higher than any other λ_j's absolute value.

Let Y_0 be the non-zero eigen vector corresponding to λ_0.

$$LY_0 = \lambda_0 Y_0 \Rightarrow (L - \lambda_0 I) Y_0 = 0$$

Or
$$\begin{pmatrix} f_1\text{-}\lambda & f_2 & f_3\ldots\ldots & f_{m-1} & f_m \\ P_1 & \text{-}\lambda_0 & 0\ldots\ldots. & 0 & 0 \\ 0 & P_2 & \text{-}\lambda_0\ldots.. & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0\ . & 0\ .\ .\ .\ldots P_{m-1} & & \lambda_0 \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ . \\ . \\ . \\ y_m(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

By operating last row, we get $P_{m-1}y_{m-1}(0) - \lambda_0 y_m(0) = 0$ and,
$$y_m(0) = \frac{P_{m-1}y_{m-1}(0)}{\lambda_0} \tag{19}$$
Without loss of generality, we may assume that $y_1(0) = 1$. Then on putting $m = 2,\ 3,\ 4\ldots m-1$, we get
$$y_2(0) = \frac{P_1}{\lambda_0}y_1(0) = \frac{P_1}{\lambda_0},\ y_3(0) = \frac{P_2}{\lambda_0}y_2(0) = \frac{P_1P_2}{\lambda_0^{\ 2}} \text{ and } y_4(0) = \frac{P_3}{\lambda_0}y_3(0) = \frac{P_1P_2P_3}{\lambda_0^{\ 3}}$$
On continuing in this way, we get $y_n(0) = \dfrac{P_1P_2P_3\ldots\ldots\ldots\ldots P_{n-1}}{\lambda_0^{\ n-1}}$

Therefore , we find
$$y_0 = \begin{pmatrix} 1 \\ P_1/\lambda_0 \\ P_1P_2/\lambda_0^{\ 2} \\ P_1P_2P_3/\lambda_0^{\ 3} \\ \vdots \\ P_1P_2P_3\cdots\cdots P_m/\lambda_0^{\ m-1} \end{pmatrix} \tag{20}$$

Hence, the elements of Y_0 the eigen vector corresponding to λ_0 are all positive.

Some Results from Leslie Matrix Model

Result I: From above theorem, we noted that all the elements of the eigen vector corresponding to the dominant eigen value λ_0 of Leslie matrix L are positive, but all eigen values of L other than λ_0 are negative real or complex so that the eigen vectors corresponding to any of these eigen values cannot have all elements positive.

Result II: The characteristics equation of the Leslie matrix is given by
$$\phi(\lambda) = \lambda^m - f_1\lambda^{m-1} - f_2P_1\lambda^{m-2} - f_3P_1P_2\lambda^{m-3}\ldots\ldots. - f_mP_1P_2\ldots\ldots.P_{m-1} = 0$$

Differentiating both sides w.r.t. λ , we get

$$\frac{d\phi}{d\lambda} = m\lambda^{m} - (m-1)f_1\lambda^{m-2} - (m-2)f_2P_1\lambda^{m-3} - (m-3)f_3P_1P_2\lambda^{m-4}.......-f_{m-1}P_1P_2.......P_{m-2}$$

$$= m\lambda^{m-1} - mf_1\lambda^{m-2} - mf_2P_1\lambda^{m-3}............-mf_{m-1}P_1P_2.......P_{m-2}$$

$$+f_1\lambda^{m-2} + 2f_2P_1\lambda^{m-3}......+(m-1)f_{m-1}P_1P_2.......P_{m-2}$$

$$=\frac{m}{\lambda}[\lambda^{m} - f_1\lambda^{m-1} - f_2P_1\lambda^{m-2}............-f_{m-1}P_1P_2.......P_{m-2}\lambda]$$

$$+\frac{1}{\lambda}[f_1\lambda^{m-1} + 2f_2P_1\lambda^{m-2}.......(m-1)f_{m-1}P_1P_2.......P_{m-2}\lambda]$$

$$\frac{d\phi}{d\lambda} = \frac{m}{\lambda}[\lambda^{m} - f_1\lambda^{m-1} - f_2P_1\lambda^{m-2}............-f_{m-1}P_1P_2.......P_{m-2}\lambda - f_mP_1P_2.......P_{m-1}]$$

$$+\frac{1}{\lambda}[f_1\lambda^{m-1} + 2f_2P_1\lambda^{m-2}.......+(m-1)f_{m-1}P_1P_2.......P_{m-2}\lambda + mf_mP_1P_2.......P_{m-1}] \tag{21}$$

Using equation (19) and (21)

$$\left(\frac{d\phi}{d\lambda}\right)_{\lambda=0} < 0, \quad \left(\frac{d\phi}{d\lambda}\right)_{\lambda=\lambda_0} > 0 \text{ and } \left(\frac{d\phi}{d\lambda}\right)_{\lambda=\infty} > 0$$

Thus, we say that $\frac{d\phi}{d\lambda}$=0 has at least one root λ_1 (say) between 0 and λ_0. Similarly we find that $\frac{d^2\phi}{d\lambda^2} = 0$ has at least one root λ_2 (say) between 0 and λ_1, this also gives that λ_2 is point of inflexion of $\phi(\lambda)$, again $\frac{d^3\phi}{d\lambda^3} = 0$ for some λ_3 lying between 0 and λ_2, and so on.

Hence, the graph of $\phi(\lambda)$ crosses the λ-axis at λ_0 and has a minimum at λ_1 and has a point of inflexion at λ_2, similarly, the graph of $\frac{d\phi}{d\lambda}$ crosses the $\lambda-$axis at λ_1 and has a minimum value at λ_2 and has a point of inflexion at λ_3, and so on.

Also, if n is even, then $\phi(0) < 0$ and $\phi(-\lambda_0) > 0$ and $\phi(\lambda) \to \infty$ as $\lambda \to -\infty$, therefore the graph of $\phi(\lambda)$ cuts the negative $\lambda-$axis at least once between 0 and $-\lambda_0$, and if m is odd, the $\phi(0) < 0$ and $\phi(-\lambda_0) > 0$ and $\phi(\lambda) \to \infty$ as $\lambda \to -\infty$, thus is this case the graph of $\phi(\lambda)$ does not cut the negative $\lambda-$axis between 0 and $-\lambda_0$. Hence from this discussion, we conclude that the graph of $\phi(\lambda)$ cuts even number of points on negative $\lambda-$axis 0 and $-\lambda_0$.

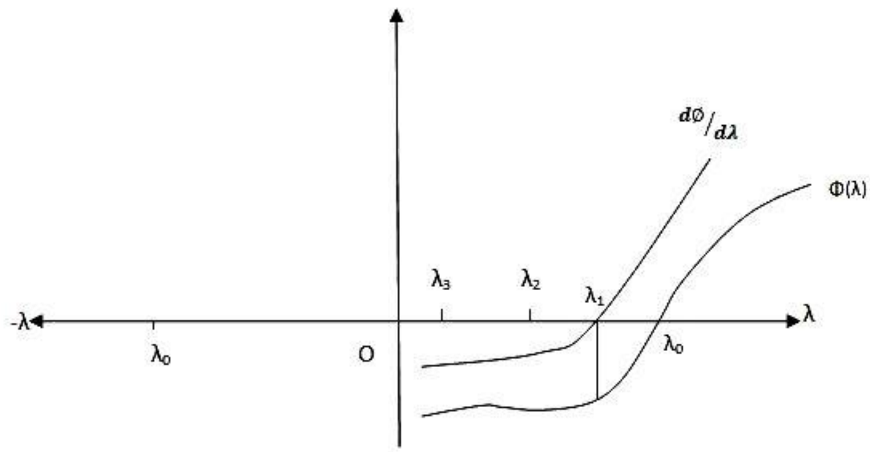

Fig. 2: Graph of $\phi(\lambda)$ and $d\phi/d\lambda$ when $\lambda>0$

Result III: If $\phi(1) < 0$ then $\lambda_0 > 1$ i.e., if $f_1 + f_2P_1 + ... f_mP_1P_2......P_{m-1} > 1$, then the populations of all age groups increase.
Result IV: If $\phi(1) > 0$ then $\lambda_0 < 1$ i.e., if $f_1 + f_2P_1 + ... f_mP_1P_2......P_{m-1} < 1$, then the populations of all age groups decrease to extinction.
Result V : If $\phi(1) = 0$ then $\lambda_0 = 1$ i.e., if $f_1 + f_2P_1 + ... f_mP_1P_2......P_{m-1} = 1$,then the population of all age group will asymptotically stationary.
Result VI : Since $\frac{d\phi}{d\lambda} > 0$ at $\lambda = \lambda_0$, this shows that λ_0 is a root of algebraic multiplicity one, therefore, if we make a small change in P_a and f_a , the corresponding change in the behaviour of the model will be determined by λ_0 .that is, a small change in P_a and f_a gives the small change is λ_0 .
Result VII: If kth age group is the last reproductive group, then we have $f_{k+1} = 0 = f_{k+2} = \cdots = f_m$
Therefore, the equation (14) becomes

$$\phi(\lambda) = \lambda^{m-k}(\lambda^k - f_1\lambda^{k-1} - f_2P_1\lambda^{k-2} - f_3P_1P_2\lambda^{k-3} \cdots - f_kP_1P_2\cdots P_{k-1} = 0$$

In this case the dominant eigen value is the dominant eigen value of the matrix.

$$\begin{pmatrix} f_1 & f_2 & \ldots & f_{k-1} & f_k \\ P_1 & 0 & \ldots & 0 & 0 \\ 0 & P_2 & \ldots & 0 & 0 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ 0 & 0 & \ldots & P_{k-1} & 0 \end{pmatrix}$$

Hence, we conclude that the post-reproduction age groups do not play any role for further growth.

Stable Age Structure in Leslie Matrix

The ratios of the population in different age-groups define the age structure, if this age structure remains constant over time, it is referred to as a stable age structure (15), that is, if $y_k(t),(k=1,2,3...m)$ is the population of various age-group, then for stable-age-structure we must have $\frac{y_i(t)}{y_j(t)}$ not a function of t.

The age-structure is asymptotically stable if $\frac{y_i(t)}{y_j(t)} \rightarrow$ finite value as $t \rightarrow \infty$.

Therefore, if a population has a stable-age-structure, then the population of different age-groups will increase in a constant ratio, so that if $Y(t)$ be the population size at time t, then $Y(t+1)$ is obtained by $\lambda Y(t)$ i.e. $\mathrm{Y}(t+1)=\lambda Y(t)$. But we have

$$Y(t+1)=LY(t) \text{ and } LY(t)=\lambda Y(t)$$

This equation shows that λ be an eigenvalue of matrix L, but the only eigenvalue of L is λ_0 for which all the elements of Y_0 are positive. Thus, if we take the initial population to be Y_0 which is given by (20), then the age distribution will never change, regardless of how much each age-group's population grows or shrinks according to $\lambda_0>1$ or $\lambda_0<1$.

However, if the initial population is different from Y_0, the population structure will continue to change over time. If the population distribution or structure becomes closer to a stable state value as time progresses ($t \rightarrow \infty$), then Y_0 represents the asymptotic-stable-structure and λ_0 indicates the rate at which the population of each group will eventually increase or decrease per unit time.

Numerical Analysis and Interpretation

For the numerical analysis of Leslie Matrix, it is necessary to estimate the life table survival probabilities (or survival rate), and this ratio is estimated using following formula:

$$\text{5 year Survival Rate } = P_{a+5} = \frac{{}_5L_{a+5}}{{}_5L_a}$$

Now for the older age group 85+ we taking survival rate is zero.

This study examines the female population data in India from 2001-2011. The analysis focuses on females aged 15 to 49, as this range represent the typical age of fertility. The data is divided into five-year age-groups, resulting in 18 age-groups for analysis. Key parameters such as the average numbers of survivor (S_a), in an age-interval, survival probabilities (P_a), and age-specific fertility rates (ASFR) will be evaluated. The Leslie Matrix is employed to project the future female population.

Table 1. Age Structured Female Population, Survival Rate and Fertility Rate

Age Groups	Number of Females 2001	Number of Females 2011	S_a	P_a	ASFR
0-4	53460857	67297225	475632	0.943998722	**0**
5-9	61735896	57399177	448996	0.991955385	**0**
10-14	59362001	50060993	445384	1.992224687	**0**
15-19	46391577	60763100	451921	0.989045553	0.0489
20-24	43551578	58255221	437080	0.986693511	0.2159
25-29	41969498	45272836	431264	1.986711156	0.1773
30-34	37004399	42401010	445533	0.986597984	0.0985
35-39	34621687	40856771	419830	0.984405593	0.0499
40-44	25924224	35939138	413283	1.980553277	0.0212
45-49	22597437	33419003	418246	0.969201423	0.0073
50-54	16777787	24637180	392765	0.948977124	**0**
55-59	14105497	20783992	372725	0.925148568	**0**
60-64	13965254	14729971	344826	0.880667351	**0**
65-69	10360686	11492427	303677	0.821392466	**0**
70-74	7198907	10102095	249438	0.746830876	**0**
75-79	3296235	6355672	186288	0.669404363	**0**
80-84	2313752	3598963	124702	0.965982903	**0**
85+	1816284	2081255	120460	**0**	**0**
Total	**496453556**	**585446029**	**6439049**		

To determine the female population numbers using the Leslie matrix methods, which takes into account age-specific fertility rates and survivorship, we apply the Leslie Matrix L based on tabular data

$$L = \begin{bmatrix}
0 & 0 & 0 & 0.0489 & 0.2159 & 0.1773 & 0.0985 & 0.0499 & 0.0212 & 0.0073 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.9440 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.9920 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.9922 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.9890 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.9867 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.9867 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.9866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9844 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.9806 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9692 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9490 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9251 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8807 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8214 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7468 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6694 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9660 & 0
\end{bmatrix}$$

To calculate the dominant Eigen value of the above Leslie matrix L, we solved above matrix in MATLAB the find out the value of $\lambda = 1.1148$. The calculated dominant eigen value is greater than one and female population is increasing with average natural increment of 1.08 percent annually. Now to project the female population for the future years recalling equation (8) i.e., $Y(t) = L^t Y(0)$ and taking census population of 2011 as the initial population vector $Y(0)$ then after matrix calculations we find out the projected female population for the years 2011 to 2051 with 5 years' time interval is given below in tabular form

Table 2. Projected Indian Female Populations Using Leslie Matrix Model

Age Gro	Projected Female's Population								
	2011	2016	2021	2026	2031	2036	2041	2046	2051
0-4	672972	742325	765908	769305	801092	857390	916036	960532	997292
5-9	573991	635284	700754	723016	726223	756230	809375	864737	906741
10-	500609	569374	630174	695117	717199	720381	750147	802864	857781
15-	607631	496717	564947	625274	689712	711623	714780	744314	796621
20-	582552	600974	491276	558758	618424	682157	703827	706950	736160
25-	452728	574800	592977	484739	551323	610195	673080	694462	697543
30-	424010	446712	567162	585097	478297	543996	602087	664135	685233
35-	408567	418327	440725	559560	577256	471887	536706	594017	655234
40-	359391	402196	411803	433852	550834	568254	464528	528336	584754
45-	334190	352402	394374	403795	425415	540122	557203	455494	518062
50-	246371	323897	341548	382228	391359	412313	523487	540042	441466
55-	207839	233801	307371	324122	362726	371391	391275	496778	512488
60-	147299	192282	216300	284364	299861	335575	343591	361988	459593
65-	114924	129722	169337	190489	250430	264077	295530	302590	318791
70-	101020	943979	106552	139092	156466	205701	216911	242746	248545
75-	635567	754455	704992	795768	103878	116853	153624	161996	181290
80-	359896	425451	505035	471925	532690	695366	782224	102836	108441
85+	208125	279096	346178	418243	437383	476644	575866	667296	833161
Tota	**585446**	**635911**	**685683**	**732740**	**779751**	**828535**	**878800**	**929155**	**978935**

Table 3. Female Population Percentage Distribution of the year 2016

Age Group	Leslie Matrix Model		Census Report-2016 (Registrar General of India)	
	Female Population (2016)	Percentage distribution	Female Population (2016)	Percentage distribution
0-4	74232580	11.7	52306115	8.2

5-9	63528494	10.0	56133391	8.8
10-14	56937423	9.0	59960668	9.4
15-19	49671753	7.8	63150065	9.9
20-24	60097474	9.5	68253101	10.7
25-29	57480049	9.0	62512186	9.8
30-34	44671213	7.0	51030356	8.0
35-39	41832751	6.6	45927320	7.2
40-44	40219634	6.3	38910646	6.1
45-49	35240239	5.5	34445490	5.4
50-54	32389746	5.1	27428816	4.3
55-59	23380121	3.7	23601540	3.7
60-64	19228281	3.0	19774263	3.1
65-69	12972205	2.0	14033348	2.2
70-74	9439793	1.5	9568192	1.5
75-79	7544556	1.2	5740915	0.9
80-84	4254515	0.7	3189397	0.5
85+	2790967	0.4	1913638	0.3
Total	**635911794**	**100**	**637879447**	**100**
Mean Absolute Percentage Error: 0.31%				

The above table shows a comparison between the female population data calculated by Leslie matrix and data given by Registrar General of India, Govt. of India for the year 2016. The mean absolute percentage error is 0.31% which is acceptable for a good projection. For the more comparative, we showed this data graphically below in the figures 3 and 4.

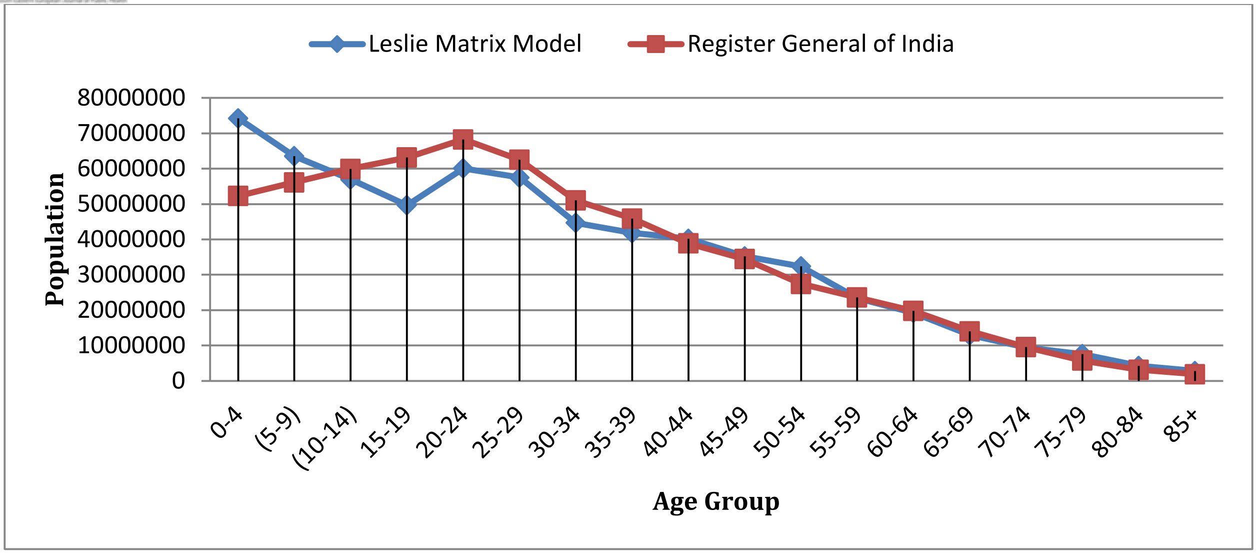

Fig. 3: Female Populations by Leslie Matrix and R.G.I

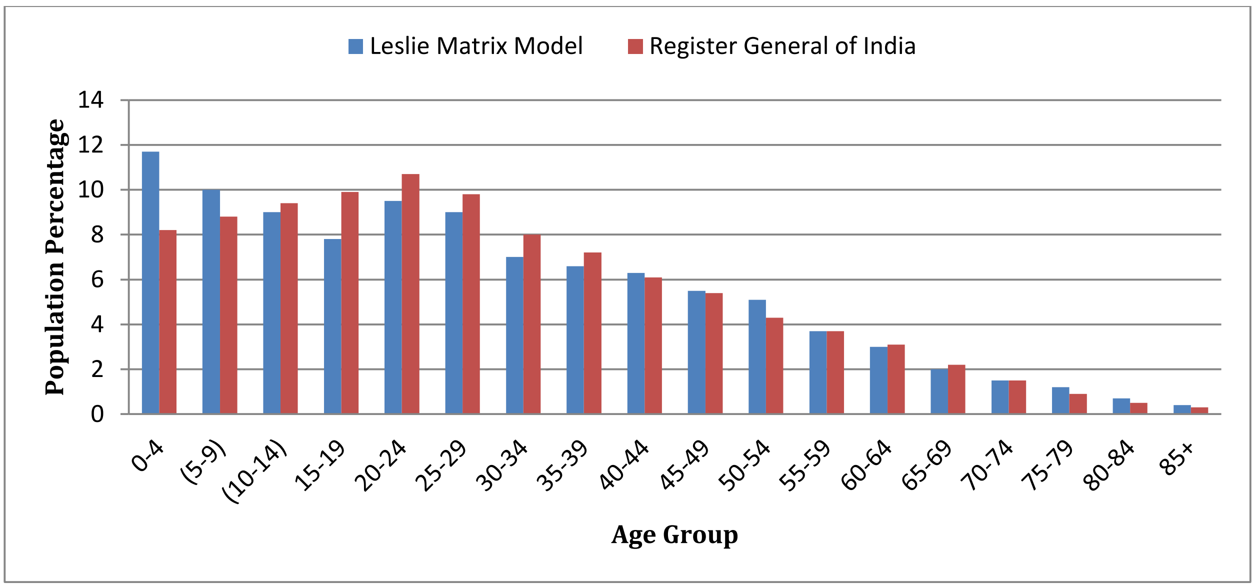

Fig. 4: Population Percentage Distributions in Age-Group

Conclusion

The primary objective of this work is to examine the properties and application of the Leslie matrix model, dealing how to construct the Leslie population projection matrix and providing a comprehensive understanding of the mathematical techniques that underpin the parameters of this matrix. Our analysis demonstrate that the Leslie matrix is governed solely by age-dependent fertility and survival rates, which are the key vital rates. Essential estimations were made to predict future population trends. The population dynamics are influenced by the dominant eigenvalue λ_0 of the Leslie matrix. If $\lambda_0 = 1$, the population remains stationary; if $\lambda_0 > 1$, the population is growing; and if $\lambda_0 < 1$, the population is declining. We identified all age-groups along with their specific vital rates, assuming that individuals within the same age-group have equal probabilities of surviving to the next age-group. The population projection reveals that the population will either increase or decrease exponentially, with all age classes growing or shrinking at the same rate. A Leslie matrix constructed using census data, incorporating calculated fertility and survival rates to project the female population in India from 2011 to 2051, with five-years age intervals. To validate the accuracy of this model, we compared our projection for 2016 with data provided by Registrar General of India, Govt. of India. The mean absolute percentage error between our computed population and official data is 0.31%, indicating both the model's general suitability and its accuracy for projecting population. Analysis shows that variation in the entries of the constructed matrix significantly impact the dominant eigenvalue and the overall population dynamics. Based on numerical results, we find that the Leslie matrix model is most effective for middle-aged groups, specially between 16 and 59 years. However , for those over 60 years and those in the 0-15 age range, it is less reliable. Additionally, this method is not suitable for the population experiencing significant migration in or out.

References

1. Leslie Ph. Some Further Notes on The Use of Matrices in Population Mathematics. Biometrika. 1948;35(3–4).

2. Montshiwa MI, Sciences M, Sherwell D. Leslie Matrix Model in Population Dynamics. Sciences-New York. 2007;

3. Weckerly FW. Matrix Population Models: Construction Analysis and Interpretation. Nat Resour Model. 2001;14(4).

4. Vindenes Y, Le Coeur C, Caswell H. Introduction to matrix population models. Demographic Methods across the Tree of Life. 2021 Jan 1;163–79.

5. Patty D, Leleury ZA, Popla JA. Leslie Matrix Construction and Its Properties in Predicting the Size of The Female Population. Barekeng: Jurnal Ilmu Matematika dan Terapan. 2024;18(1).

6. Jones R, White K. Stochastic Leslie Matrix Models in Demographic Forecasting. Popul Stud (NY). 2022;76(1):67–82.

7. Anderson J, Brown T, Williams P. Age-Structured Population Projections Using the Leslie Matrix. Demographic Research Journal. 2022;4(28):341–58.

8. Doak DF, Waddle E, Langendorf RE, Louthan AM, Isabelle Chardon N, Dibner RR, et al. A critical comparison of integral projection and matrix projection models for demographic analysis. Ecol Monogr. 2021 May 4;91(2).

9. Davis A, Kessler B. Density-dependent Leslie matrix modeling for logistic populations with steady-state distribution control. Mathematical Scientist. 2016;41(2).

10. Liu Y, Zhao Q. The Impact of the Two-Child Policy on Population Dynamics in China: A Leslie Matrix Approach. . Asian Popul Stud. 2023;19(2):199–214.

11. Caswell H. Perturbation analysis of nonlinear matrix population models. Demogr Res. 2008 Mar 12;18:59–116.

12. McElderry RM, Gaoue OG. Pitfalls to avoid in nonlinear perturbation analysis of structured population models. Popul Ecol. 2022;64(4).

13. Ezard THG, Bullock JM, Dalgleish HJ, Millon A, Pelletier F, Ozgul A, et al. Matrix models for a changeable world: The importance of transient dynamics in population management. Journal of Applied Ecology. 2010;47(3).

14. Stott I, Townley S, Hodgson DJ. A framework for studying transient dynamics of population projection matrix models. Vol. 14, Ecology Letters. 2011.

15. Bukkuri A, Brown JS. Integrating eco-evolutionary dynamics into matrix population models for structured populations: Discrete and continuous frameworks. Methods Ecol Evol. 2023;14(6).

16. Akamine T, Suda M. The Growth Rates of Population Projection Matrix Models in Random Environments. Aqua-BioScience Monographs. 2011;4(3).

17. López I, Varga Z, Gámez M, Garay J. Monitoring in a Discrete-Time Nonlinear Age-Structured Population Model with Changing Environment. Mathematics. 2022;10(15).

18. Cáceres MO, Cáceres-Saez I. Calculating effective growth rate from a random Leslie model: Application to incidental mortality analysis. Ecol Modell. 2013;251.