

t-CF pebbling number of some path related graphs

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KEYWORDS ABSTRACT

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Graph pebbling is a mathematical process involving the movement of pebbles on a connected graph G according to specific rules. Assume G is a graph with some pebbles distributed over its vertices. A CF pebbling move is defined as the removal of x pebbles from one vertex, followed by discarding $\lfloor \frac{x}{2} \rfloor$ pebbles and moving the remaining $\lceil \frac{x}{2} \rceil$ pebbles to an adjacent vertex. The t -CF pebbling number, $\lambda_t(G)$, of a connected graph G , is the least positive integer n such that any distribution of n pebbles on G allows t pebbles to be carried to any arbitrary vertex using a sequence of CF pebbling moves. In this paper, we determine the t -CF pebbling number of paths and path-related graphs, providing insights into the relationship between graph structure and pebbling strategies.

1 Introduction

Pebbling in Graphs was first studied by Chung[1]. Pebbling numbers are a central concept in graph theory, focusing on the movement and placement of resources (pebbles) on graph vertices under specific rules. The t -ceiling floor (t -CF) pebbling number is a variation that imposes unique constraints, combining aspects of resource management with vertex-specific requirements.

The study of t -CF pebbling number of paths and path-related graphs is particularly interesting due to their simple, linear structure and relevance in real-world applications such as network routing, resource distribution, and data flows. These graphs provide a foundation for analyzing pebbling strategies and understanding how constraints influence resource allocation. In this paper, we study the t -CF pebbling number of path-related graphs, analyzing how this number evolves with varying graph configurations. We present precise formulations for the t -CF pebbling number of path-related graphs. Here, $p(v)$ denotes the number of pebbles placed on the vertex v in a graph G and $t > 1$.

2 Preliminaries

Definition 2.1. [3] Assume G is a graph with some pebbles distributed over its vertices. A CF pebbling move is when x pebbles are removed from one vertex, $\lfloor \frac{x}{2} \rfloor$ pebbles are thrown away, and $\lceil \frac{x}{2} \rceil$ pebbles are moved to an adjacent vertex.

Definition 2.2. [3] A CF pebbling number $\lambda(G, v)$ of a vertex v of a graph G is the smallest number $\lambda(G, v)$ such that at least one pebble may be moved to the target vertex v using a sequence of CF pebbling moves, for any placement of $\lambda(G, v)$ pebbles on the vertices of G . The maximum $\lambda(G, v)$ over all the vertices of G is the CF pebbling number of a graph, denoted as $\lambda(G)$.

3 The t-CF pebbling number of some path related graphs

Definition 3.1. Let G be a graph with some pebbles distributed over its vertices. The t -CF pebbling number $\lambda_t(G)$ of a connected graph G is the least positive integer n such that any

distribution of n pebbles on G allows t pebbles to be carried to any arbitrary vertex using a sequence of CF pebbling moves.

Theorem 3.1. The 2-CF pebbling number of $P_n \times K_2$ is $\lambda_2(P_n \times K_2) = 2^{n+1} + 1$

Proof. Let $V(P_n \times K_2) = \{v_1, v_2, \dots, v_n, v_{n+1}, u_1, u_2, \dots, u_{n+1}\}$ and $E(P_n \times K_2) = \{\{v_i v_{i+1}\} \cup \{u_j u_{j+1}\} \cup \{v_i u_i\} : 1 \leq i, j \leq n\}$. Let $\deg(v_1) = \deg(v_n + 1) = \deg(u_1) = \deg(u_{n+1}) = 2$ and $\deg(v_i) = 3 = \deg(u_i)$ for all $i \neq 1, n + 1$. By placing 2^{n+1} in v_{n+1} , two pebbles could not be moved to u_1 . So, $\lambda_2(P_n \times K_2) \geq 2^{n+1} + 1$.

Assume u_1 be our target vertex and $p(u_1) = 0$. If $p(u_2) \geq 3$ or $p(v_1) \geq 3$ then two pebbles could be moved to u_1 . If $p(u_2) \leq 1$ and $p(v_2) \leq 1$.

If $p(u_3) \geq 5$ or $p(v_2) \geq 5$ and $p(u_2) = 0 = p(v_1)$ then two pebbles could be moved to u_1 . If $p(u_2) = 1 = p(v_2)$ and $p(u_3) = 2 = p(v_3)$ then two pebbles could be moved to u_1 . Assume If $p(u_3) \geq 7$ or $p(u_2) \geq 7$ then two pebbles could be moved to u_1 . Proceeding like this, $2^{n+1} + 1$ pebbles are placed in v_{n+1} and hence two pebbles could be moved to u , and we are done. Thus, $\lambda_2(P_n \times K_2) \leq 2^{n+1} + 1$. Hence $\lambda_2(P_n \times K_2) = 2^{n+1} + 1$.

Theorem 3.2. The t -CF pebbling number of $P_n \times K_2$ is $\lambda_t(P_n \times K_2) = 2^{n+1}(t - 1) + 1 \forall t > 1, \forall n \geq 2$.

Proof. Let u be any target vertex. By placing $2^{n+1}(t - 1)$ pebbles on v_{n+1} , t pebbles could not be moved to u . So, $\lambda_t(P_n \times K_2) \geq 2^{n+1}(t - 1) + 1$. When $t=2$, using $2^{n+1} + 1$ pebbles two pebbles could be moved to any target vertex u , by theorem 3.1.

Assume $t > 2$, there are $2^{n+2} + 1$ pebbles, using 2^{n+2} pebbles two pebbles could be moved to our target vertex u , by the remaining $2^{n+1}(t - 1) + 1 - 2 = 2^{n+1}(t - 1 - 2) + 1 = 2^{n+1}(t - 3) + 1$ pebbles, $t-2$ pebbles could be moved to the target vertex and hence by induction, the pebbles could be moved to u .

Definition 3.2. [2] A graph that joins the empty graph K_m on m nodes and the path graph P_n on n nodes is called a fan graph. If $m = 1$, it is called a fan graph, and if $m = 2$, it is called a double fan.

Theorem 3.3. The 2-CF pebbling number of $P_n + 2K_1$ is $\lambda_2(P_n + 2K_1) = n + 4$.

Proof. Let $G = P_n + 2K_1$ and $V(G) = \{v_0, v_0', v_1, \dots, v_{n+1}\}$. Assume v_0 is our target vertex.

By placing 2 pebbles on v_0' and placing a single pebble on each of the vertices $v_1, v_2, \dots, v_{n+1}$, two pebbles cannot be moved to the target vertex v_0 . So $\lambda_2(P_n + 2K_1) \geq 3 + n + 1 = n + 4$. Distributing $n + 4$ pebbles on all vertices of $P_n + 2K_1$, assume v_1 is our target vertex and $p(v_1) = 0$.

If $p(v_0) \geq 3$, or $p(v_2) \geq 3$, or $p(v_0') \geq 3$, then two pebbles can be moved to v_1 . If $p(v_0) = 2$ and $p(v_2) = 2$, or $p(v_0) = 2$ and $p(v_0') = 2$, or $p(v_0') = 2$ and $p(v_2) = 2$, then two pebbles can be moved to v_1 .

Assume $p(v_0) = 1$, $p(v_2) = 1$, and $p(v_0') = 2$. If $p(v_3) \geq 2$, then from v_3 , a pebble can be moved to v_0' , and from v_0' , two pebbles can be moved to v_1 . Now assume $p(v_0) = 1$, $p(v_2) = 1$, $p(v_0') = 1$, and $p(v_3) = 1$. If $p(v_4) \geq 3$, then from v_4 , two pebbles can be moved to v_0 , and from v_0 , two pebbles can be moved to v_1 . If $p(v_4) = 2$, $p(v_0) = 1$, $p(v_2) = 1$, $p(v_3) = 1$, $p(v_0') = 1$, and $p(v_5) \geq 2$, then we are done.

Assume $p(v_0) = p(v_0') = p(v_3) = p(v_2) = \dots = p(v_n) = 1$, then the remaining $n + 4 - (n + 1) = 3$ pebbles are placed on v_{n+1} , and from v_{n+1} , two pebbles can be moved to v_0 , and from v_0 , two pebbles can be moved to v_1 .

Assume $p(v_0) = 0 = p(v_0')$ and $p(v_2) = p(v_3) = \dots = p(v_n) = 1$, then the remaining five pebbles are placed on v_{n+1} , and from v_{n+1} , three pebbles can be moved to v_0 , and two pebbles can be moved to v_1 .

Assume $p(v_0) = p(v_0') = p(v_3) = p(v_2) = \dots = p(v_n) = 0$, then all $n + 4$ pebbles are placed on v_{n+1} , and from v_{n+1} , $\left\lceil \frac{n+4}{2} \right\rceil$ pebbles are moved to v_0 . Since $n \geq 2$, we have:

$$n + 4 \geq 6 \quad \text{and} \quad \frac{n+4}{2} \geq 3$$

From v_0 , two pebbles can be moved to v_1 , and we are done.

Theorem 3.4. The t -Cf pebbling number of $P_n + 2K_1$ is $\lambda_t(P_n + 2K_1) = 4(t-1) + n$

$\forall n \geq 2, t \geq 2$.

Proof. By placing $4t-4$ pebbles at v_i , and place a single pebble on each of the vertices of $P_n + 2K_1 \setminus \{v_0, v_1, v_i\}$, t pebbles cannot be moved to v_1 . So $\lambda_t(P_n + 2K_1) \geq 4(t-1) + n$. Distributing $4(t-1) + n$ pebbles on all the vertices of $P_n + 2K_1$, when $t = 2$, result is true by theorem 3.3.

Assume $t > 2$, there are atleast $n + 8$ pebbles on $P_n + 2K_1$, using atmost eight pebbles, two pebbles could be moved to v_1 , using the remaining $4(t-1) + n - 8 = 4(t-3) + n$ pebbles, $t-2$ pebbles could be moved to v_1 . So $\lambda_t(P_n + 2K_1) \geq 4(t-1) + n$.

Theorem 3.5. The 2-CF pebbling number of the graph $P_n \odot K_1$ is

$$\lambda_2(P_n \odot K_1) = 2^{n+2} + n$$

Proof. Let $V(P_n \odot K_1) = \{v_1, v_2, \dots, v_{n+1}, v_1', \dots, v_{n+1}'\}$. Let v_{n+1}' be the target vertex. Now, place 2^{n+2} pebbles on v_1' and one pebble on each of the $n-1$ pendant vertices except v_{n+1}' . Thus, a pebble cannot be moved to the target vertex. Hence $\lambda_2(P_n \odot K_1) \geq 2^{n+2} + n$. Now place $2^{n+2} + n$ pebbles on the vertices of the graph.

Case (i): Choose any vertex v_i , $1 \leq i \leq n+1$, such that $p(v_i) = 0$ as a target vertex. Consider the path $P' = \{v_1', v_1, \dots, v_{n+1}, v_{n+1}'\}$ of length $n+2$. If all $2^{n+2} + n$ pebbles are distributed on the path P' , then our target vertex can be easily pebbled with two pebbles. If all $2^{n+1} + n$ pebbles are distributed in a way that $p(P') = 0$, then at least $1 + 2^{n+1}$ pebbles will reach the intermediate vertices v_j , $2 \leq j \leq n$, $i \neq j$, of P' .

Thus, our target vertex can be pebbled with two pebbles as the distance between the target vertex and the intermediate vertices v_j , $2 \leq j \leq n$, $i \neq j$, of P' is at most $n-1$. If $2^{n+2} + n$ pebbles are distributed on both P_n and the pendant vertices, let $s > 0$ be the number of pebbles distributed on the pendant vertices such that $p(P_n) = 2^{n+2} + n - s$.

Subcase 1: If $s = 1$, then $2^{n+2} + n - 1$ pebbles on P_n are enough to reach the target vertex with two pebbles.

Subcase 2: Suppose $n > s > 1$. Then the number of pebbles on the path P_n will be at least $2^{n+2} + 1$. Hence, two pebbles can reach the target vertex.

Subcase 3: Suppose $s \geq n$. Then the number of pebbles that the path P_n can receive from both the path P_n and the pendant vertices will be at least $1 + 2^{n+1}$. Since P_n is a path of length n , two pebbles can be moved to the target vertex.

Case (ii): Let v_i' be the target vertex such that $p(v_i') = 0$ and $1 \leq i \leq n+1$.

If all the pebbles are placed either on the path P_n or on the pendant vertices except the target vertex, then the target vertex can be easily pebbled as the distance between the target vertex and any other vertex in the graph is at most $n+2$.

If $2^{n+2} + n$ pebbles are distributed on both P_n and the pendant vertices, let $s > 0$ be the number of pebbles distributed on the pendant vertices such that $p(P_n) = 2^{n+2} + n - s$.

Now, proceeding as in Case (i), for any values of s , the target vertex can be easily pebbled with two pebbles. Since each pendant vertex is non-adjacent, an intermediate vertex of the path P_n at a distance of i from the initial vertex of P_n having one pebble is equivalent to placing $1 + 2^i$ pebbles on the initial vertex of P_n . Moreover, the distance between the target vertex and any other vertex in the graph is at most $n+2$.

Theorem 3.6. The t -CF pebbling number of the graph $P_n \odot K_1$ is

$$\lambda_t(P_n \odot K_1) = 2^{n+2}(t-1) + n$$

Proof. We will prove the result by induction on t . For $t=2$, the result is true by theorem 3.5. Now assume that the result is true for all values less than t . So using $2^{n+2}(t-2) + n$ pebbles $t-1$ pebbles could be moved to the target vertex. Now with the remaining 2^{n+2} pebbles one pebble could be moved to the target vertex since the distance between the target vertex and any other vertex in the graph is less than $n+2$.

Theorem 3.7. The t -CF pebbling number of $S_p(P_m, K_{1,n})$ is $2^{m+1}(t-1) + n$.

Proof. Let $V(S_p(P_m, K_{1,n})) = \{v_1, v_2, \dots, v_{m+1}\} \cup \{y_1, y_2, \dots, y_n\}$ where $(v_1, y_1, y_2, \dots, y_n)$ are of

degree 1, and $E(S_p(P_m, K_{1,n})) = \{\{v_i v_{i+1}\} \cup \{v_{m+1} y_k\} : 1 \leq i \leq m, 1 \leq k \leq n\}$.

Let us assume that $t = 2$.

Let y_1 be our target vertex. Consider the following distribution of $2^{m+1} + n - 1$ pebbles on G .

(i) $p(v_1) = 2^{m+1}$

(ii) $p(y_i) = 1 \forall i \neq 1$

Since $d(v_1, y_1) = m + 1$, with the help of $1 + 2^m$ pebbles, only one can be moved to y_1 . Thus $\lambda(G) \geq 2^{m+1} + n$.

Suppose $2^{m+1} + n$ pebbles are placed on G .

Case 1: Let the $v \in V(P_m)$ be our target vertex.

Subcase 1: Let all pebbles be placed on P_m .

There is nothing to prove.

Subcase 2: $p(K_{1,n} - \{v_{m+1}\}) = 2^{m+1} + n$.

The number of pebbles that can reach v_{m+1} , with $1 + 2^m$ pebbles, two pebbles can be moved to the target vertex.

Subcase 3: $p(K_{1,n} - \{v_{m+1}\}) = 2^{m+1} + n - s$.

Then the pebbles that can reach P_m is atleast $2^m + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{s}{2} \right\rfloor + s$

$= 2^m + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor > 1 + 2^n$, since $s, n > 0$.

Thus 2 pebbles can be moved to the target vertex.

Case 2: Let $v \in V(K_{1,n}) - \{v_{m+1}\}$ be our target vertex.

Subcase 1: Let all pebbles be placed on P_m .

Then the number of pebbles that can reach v_{m+1} could be atleast 4. Thus 2 pebbles can reach the target vertex.

Subcase 2: $p(K_{1,n} - \{v_{m+1}\}) = 2^{m+1} + n$.

Clearly there exist atleast one vertex $y \in K_{1,n}$ such that $p(y) \geq 5$ or there exists atleast two vertices $x, y \in K_{1,n}$ with $p(x) \geq 2$ and $p(y) \geq 3$ or there exists atleast three vertices $x, y, z \in K_{1,n}$ with $p(x) \geq 2, p(y) \geq 2$ and $p(z) \geq 2$. Thus 2 pebbles can reach the target vertex.

Subcase 3: If s pebbles are placed on $K_{1,n} - \{v_{m+1}\}$ such that $s > 0$. The number of pebbles on the path P_m is $2^{m+1} + n - s$. If atleast one vertex of $K_{1,n} - \{v_{m+1}\}$ has atleast five pebbles, then two pebbles can reach the target vertex. Let $p(v_{m+1}) = 0$. If atleast three pebbles of $K_{1,n} - \{v_{m+1}\}$ has atleast two pebbles each then two pebbles can be moved to the target vertex. Otherwise, the only possible way is atleast two vertices of $K_{1,n} - \{v_{m+1}\}$ has atleast two pebbles and the remaining vertices of $K_{1,n} - \{v_{m+1}\}$ has atleast one pebble. That is $s \leq n - 1$. Clearly atleast one pebble could be moved to v_{m+1} and hence two pebbles could be moved to the target vertex.

Let us assume that the result is true for all values less than t . Now we have to prove that the result is true for all t . Now using $2^{m+1}(t-2) + n$ pebbles we can place $t-1$ pebbles on the target vertex. Now we are left with the remaining 2^{m+1} pebbles one pebble can be moved to the target vertex, since distance between any two vertices in the graph is atleast $m+1$. Thus, t pebbles can be moved to the target vertex. Hence we are done.

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